18.085 SUMMER 2013 - QUIZ 3 - AUGUST 15, 2013

YOUR NAME: ______________________________

YOUR SCORE: ____________ **/ 100 +** __________ **/ 20 extra credit**

THE QUIZ IS OPEN BOOK, OPEN NOTES AND NO CALCULATORS

GRADING:

(1) **1.** ______________

(2) **2.** ______________ + ______________ **BONUS POINTS**

(3) **3.** ______________

(4) **4.** ______________

(1) (20 points.)
The dft of the signal x is

$$\hat{x} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

What signal is sampled in x?
Sketch the real and the imaginary part of x. (With our usual notation, here we have that N=11). **Solution:**

As

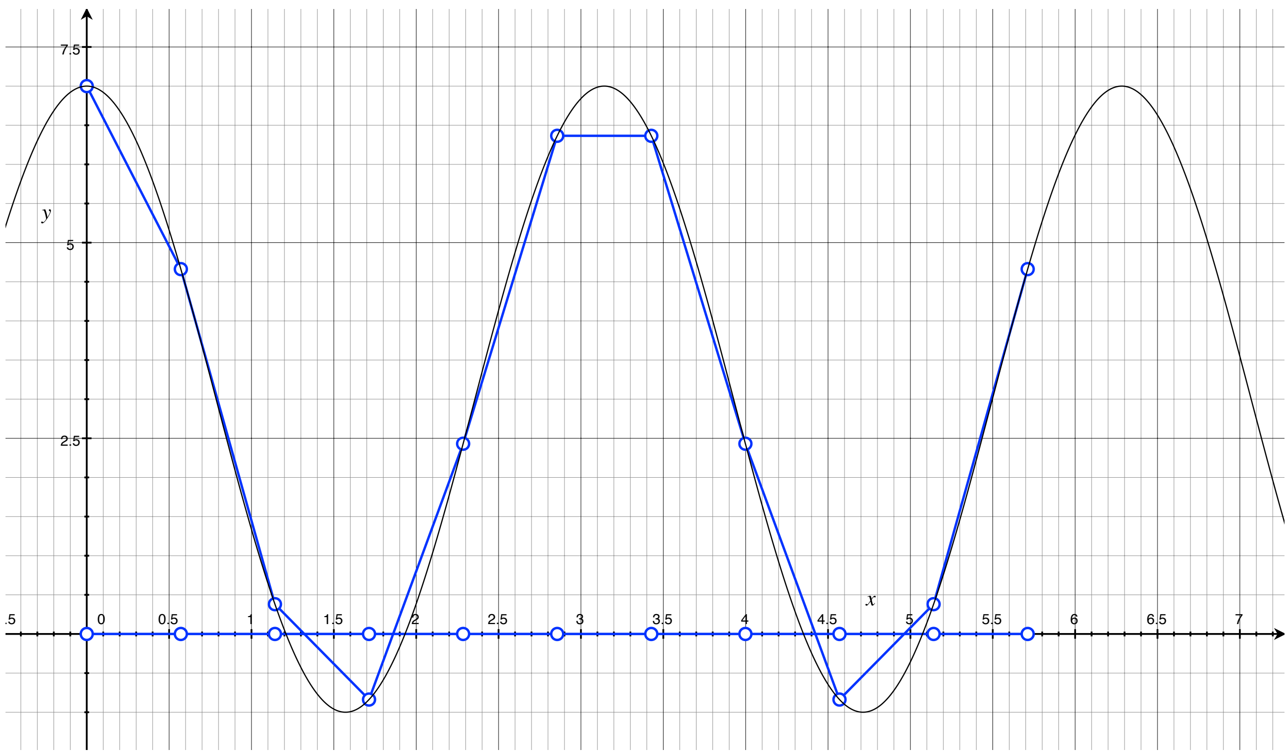
$$x = G_N \hat{x},$$

it follows that

$$x = \sum_{k=0}^{N-1} \hat{x}_k v_k = 3v_0 + 2v_2 + 2v_9 =$$

$$3\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ e^{1\cdot 2\cdot 2\pi i/11} \\ e^{2\cdot 2\cdot 2\pi i/11} \\ \vdots \\ e^{10\cdot 2\cdot 2\pi i/11} \end{pmatrix} + 2\begin{pmatrix} 1 \\ e^{-1\cdot 2\cdot 2\pi i/11} \\ e^{-2\cdot 2\cdot 2\pi i/11} \\ \vdots \\ e^{-10\cdot 2\cdot 2\pi i/11} \end{pmatrix} = \begin{pmatrix} 3+4 \\ 3+4\cos(\frac{2\cdot 2\pi}{11}) \\ 3+4\cos(\frac{4\cdot 2\pi}{11}) \\ \vdots \\ 3+4\cos(\frac{20\cdot 2\pi}{11}) \end{pmatrix}.$$

The graph of the real part is the graph of a $f(t) = 3 + 4\cos(2t)$ and the points we are sampling are those corresponding to $x = 0, \frac{2\pi}{11}, \frac{2\cdot 2\pi}{11}, \ldots, \frac{10\cdot 2\pi}{11}$.



7.5
5
2.5
y
x
.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7

(2) $(15 + 15 + 10$ points $+10 + 10$ bonus points.)
Which of these equations can be solved? If you can solve them, then show a solution, otherwise explain why it is not possible to find a solution.

a)

$$\operatorname{div}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = x^2 - 3\lambda y,$$

where λ is a fixed real parameter.

b) The Laplace equation on the unit circle, with boundary condition,

$$-\Delta u = 0, \;\; u(1,\theta) = \sin(2\theta) + \cos(6\theta) + 1.$$

c)

$$\operatorname{div}\begin{pmatrix} -\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \log x^2 - 3y,$$

Solution:

a) A solution to

$$\operatorname{div}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \Delta u = x^2 - 3\lambda y$$

is given by $u(x,y) = \frac{x^4}{12} - \lambda\frac{x^y}{2}$.

b) A solution to

$$-\Delta u = 0, \;\; u(1,\theta) = \sin(2\theta) + \cos(6\theta) + 1.$$

is given by

$$u(r,\theta) = r^2\sin(2\theta) + r^6\cos(6\theta) + 1$$

as shown in the book at page .

c) The equation

$$\operatorname{div}\begin{pmatrix} -\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \log x^2 - 3y,$$

has no solution, since

$$\operatorname{div}\begin{pmatrix} -\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x} \end{pmatrix} = -\frac{\partial^2 s}{\partial x \partial y} + \frac{\partial^2 s}{\partial y \partial x} = 0.$$

Caveat: the next two are bonus questions.

d1) Find a family of curves $s(x,y) = C$ that is everywhere orthogonal to the family of curves $u(x,y) = e^x(\cos(y))$.

d2) Can you tell for which complex function the function $u(x,y)$ in d1) is the real part?

Solution:

d1) As $\operatorname{div}(\nabla u) = e^x \cos(y) - e^x \cos(y) = 0$, then u admits a stream function s, which is determined by the two conditions

$$\begin{aligned} \frac{\partial s}{\partial y} &= \tfrac{\partial u}{\partial x} = e^x \cos(y), \\ -\frac{\partial s}{\partial x} &= \tfrac{\partial u}{\partial y} = -e^x \sin(y). \end{aligned}$$

Then,

$$s = \int \frac{\partial s}{\partial y} dy + F(x) = \int e^x \cos(y) dy + F(x) = e^x \sin(y) + F(x),$$

$$s = \int \frac{\partial s}{\partial x} dx + G(y) = \int e^x \sin(y) dx + G(y) = e^x \sin(y) + G(y).$$

Hence $s(x,y) = e^x \sin(y)$ is a solution.

d2) The complex function we are looking for has u as real part and s as imaginary part, hence

$$\begin{aligned} h(x,y) &= u(x,y) + is(x,y) = e^x \cos(y) + ie^x \sin(y) \\ &= e^x(\cos(y) + i\sin(y)) = e^z \end{aligned}$$

where $z = x + iy$.

(3) (15 points). Let $f(x) = |x| - g(x)$ on $[-\pi, \pi]$, where $g(x)$ is defined as follows,

$$g(x) = \begin{cases} 15 & \text{for } x \in [0, \pi] \\ -1 & \text{for } x \in (-\pi, 0). \end{cases}$$

Find the Fourier series of $f(x)$, when f is extended periodically to the real line, by the rule $f(2\pi + x) = f(x)$.

Solution:
We can rewrite g as $g(x) = 7 + 8SW(x)$, where

$$SW(x) = \begin{cases} 1 & \text{for } x \in [0, \pi] \\ -1 & \text{for } x \in (-\pi, 0). \end{cases}$$

The Fourier series of $SW(x)$, that we saw in class multiple times, is

$$\frac{4}{\pi} \sum_{\mathbb{N} \ni k, kodd} \frac{\sin(kx)}{k},$$

hence the Fourier series of $g(x)$ is

$$g(x) = 7 + \frac{32}{\pi} \sum_{\mathbb{N} \ni k, kodd} \frac{\sin(kx)}{k}.$$

$h(x) = |x|$ is an even function, hence its Fourier series will be a series of cosines, $\sum_{k=0}^{\infty} c_k \cos(kx)$. Let us compute it.

$$\begin{aligned} c_0 &= \tfrac{1}{2\pi} \textstyle\int_{-\pi}^{\pi} -|x| dx = \tfrac{\pi}{2} \\ c_k &= \tfrac{1}{\pi} \textstyle\int_{-\pi}^{\pi} -|x| \cos(kx) dx = \begin{cases} 0 & \text{for k even} \\ -\frac{4}{\pi k^2} & \text{for k odd.} \end{cases} \end{aligned}$$

Hence the Fourier series of $f(x)$ is

$$f(x) = \frac{\pi}{2} - 7 + \sum_{\mathbb{N} \ni k, k\text{odd}} -\frac{32 \sin(kx)}{\pi k} - \frac{4 \cos(kx)}{\pi k^2}.$$

(4) $(5+5+5+5+5$ points)

a) Consider the function $f(x) = e^{-x},\ x \in [-\pi, \pi)$ which is extended periodically to the real line, by the rule $f(2\pi + x) = f(x)$. Draw the graph in $[-2\pi, 2\pi]$.
The functions satisfies the following differential equation

$$\frac{d}{dx} f(x) + f(x) = g(x), \tag{0.1}$$

for some function $g(x) : [-\pi, \pi] \to \mathbb{R}$. Find such g.

b) Compute the complex Fourier series of $f(x)$,

$$f(x) = \sum_{k \in \mathbb{Z}} d_k e^{ikx},$$

in the standard way.

Solution:

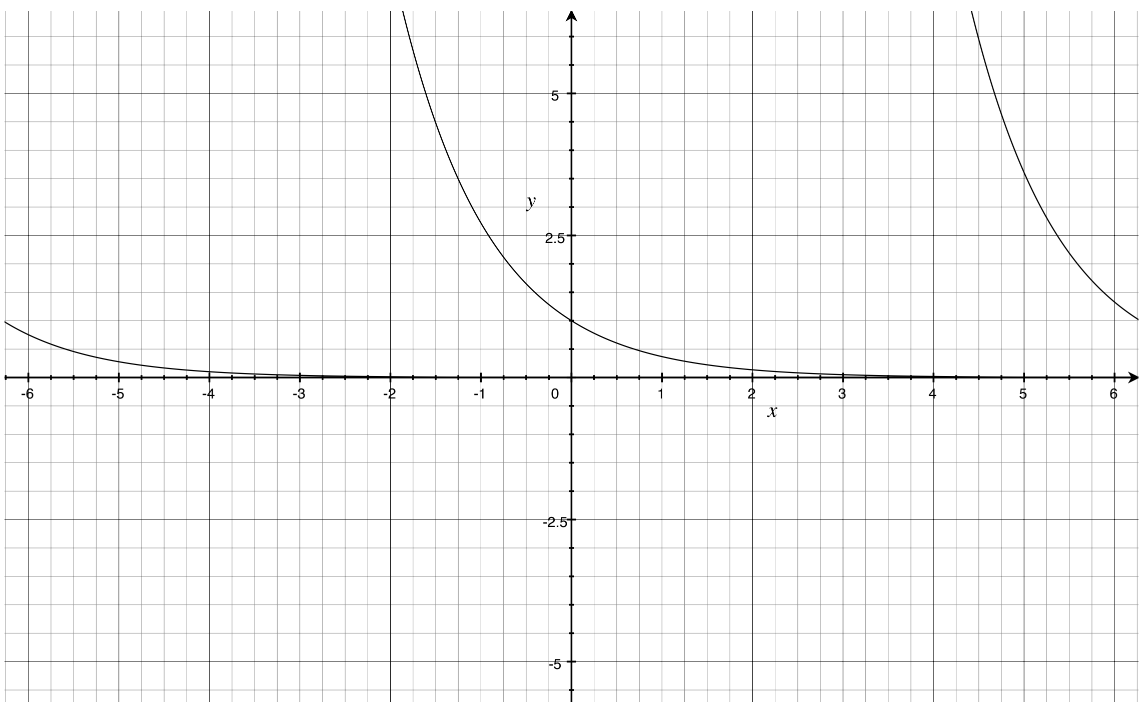


FIGURE 1. The graph of f.

a) For the differential equation, we have that

$$\frac{d}{dx} e^{-x} + e^{-x} = -e^{-x} + e^{-x} = 0,$$

hence $g(x)$ is the constantly 0 function.

b)

$$d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-ikx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(1+ik)x}dx = -\frac{1}{2\pi}\frac{e^{-(1+ik)x}}{(1+ik)}|_{-\pi}^{\pi} =$$

$$= \frac{1}{2\pi}[\frac{e^{(1+ik)\pi}}{(1+ik)} - \frac{e^{-(1+ik)\pi}}{(1+ik)}] = \frac{\sinh((1+ik)\pi)}{\pi(1+ik)}.$$

c) Try to compute the Fourier coefficients of f using the differential equation (0.1). Do you get the same result as in part b?
d) Explain the reason for the answer you gave in part c.
e) Compute

$$\sum_{k\in\mathbb{Z}} |d_k|^2.$$

Solution:

c) Applying the differential equations to the Fourier coefficients yields the following equations

$$d_k(ik) + d_k = 0, \text{ for all } k \in \mathbb{Z}, \ k \neq 0.$$

These equations imply that for $|k| \neq 1$, then $d_k = 0$, which is different from what we obtained in part b).

d) The problem is that f is not differentiable when we look at it as a function on the real line. Hence, as the differential equations (0.1) holds only in $(-\pi, \pi)$ and the discontinuity at $-\pi$ is the source of the discrepancy in the results between part b) (the right one) and part c) (the wrong one).

e)

$$\sum_{k\in\mathbb{Z}} |d_k|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2x} dx = \frac{e^{2\pi} - e^{-2\pi}}{4\pi}.$$