18.085 SUMMER 2013 - QUIZ 1 - JUNE 28, 2013

Please PRINT your name on the provided booklet.

(1) (20 points.) We want to find an approximate solution, using finite differences, for the differential equation (with boundary conditions)

$$\begin{cases} -\frac{d^2}{dx^2}u(x) = -x, \; x \in [0,1] \\ u(0) = 0 = u'(1) \end{cases} \tag{0.1}$$

Please set up the discretization problem for a mesh size of $h = 1/4$, in matrix form, that is, in the form $Av = b$. Do NOT solve the system. Your answer should contain an explanation (an equation or a few words is fine!) of how you will treat the term $-\frac{d^2u}{dx^2}$, how you will treat boundary conditions, and how you will treat the right-hand side of (0.1), which is equal to $-x$. Then give the values of the entries of matrix A and vector b, and say the meaning of vector v. Again, do NOT solve for v.

(2) (10 points.) This problem is about least-squares, or finding the best fit. We are given four (x, y) points and want to fit a parabola $y = a + bx + cx^2$ to those points, minimizing the sum of the squares of the errors. The four points are:

$$(0,1), (2,5), (1,3), (-1,-1).$$

Set up the least squares system, i.e. give the entries of the matrix A and the vector b in the normal equations $A^TAu = A^Tb$. Make sure to say what the entries of the vector u mean (but do NOT solve for u, that is, do NOT solve the system, and do NOT multiply A^T with A or A^T with b. Just give A and b, and say what u means).

(3) (20 points.) Again least-squares: you are now given the normal equations, and you need to solve them. That is, someone multiplied A^T with A and b for you and tells you

$$M = A^TA = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{pmatrix}, \qquad y = A^Tb = \begin{pmatrix} 8 \\ 14 \\ 22 \end{pmatrix}.$$

Your task is now to solve for u in the normal equations $A^TAu = A^Tb$ (or in $Mu = y$, same thing). To do this, find the LU decomposition of M and then use back substitution. Any other method to find u, including guessing, will get you 0.

(4) (20 points.) This question refers to matrices A and B, please don't confuse them!

$$A = \begin{pmatrix} -1 & 5 \\ 3 & -15 \end{pmatrix}, \qquad B = \begin{pmatrix} 15 & 5 \\ 3 & 1 \end{pmatrix}.$$

a) Find the eigenvalues and eigenvectors of the matrix A. Show your work.
b) Give an orthonormal basis for the null space of the matrix A. No need to justify.
c) Give an orthonormal basis for the null space of the matrix B. No need to justify.

(5) (20 points.) Compute the QR decomposition for the following matrix

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and use this QR decomposition to solve the linear system

$$Ax = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$$

Show your work. Do NOT use the LU decomposition, or any other method than QR, to solve this system.

(6) (10 points.) Consider the following set $\mathcal{S}$ of vectors in $\mathbb{R}^4$:

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

For these questions, you do not need to show your work, just an answer is fine.

a) Are the vectors in $\mathcal{S}$ independent? If not, find a maximal subset of vectors in $\mathcal{S}$ which are independent.

b) What is the span of the vectors in $\mathcal{S}$?