18.085, PROBLEM SET 2, SOLUTIONS

(1) The inverses are

$$\begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -l_1 & 1 & 0 \\ -l_2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -l_1 & 1 & 0 \\ -l_2 + l_1 & -1 & 1 \end{pmatrix}$$

$$, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 110 & 0 \\ 0 & 0 & 11110 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{110} & 0 \\ 0 & 0 & \frac{1}{11110} \end{pmatrix}.$$

(2) a)

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

b)

$$C = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

c)

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

d)

$$E = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ \theta \in [0, 2\pi].$$

(3) We have to split the linear system into two parts

$$\begin{cases} Ux = y \\ Ly = f \end{cases}$$

a

$$Ly = f \text{ is } \begin{pmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix} y = \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$$

whose solution is $y = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. The second part then becomes

$$\begin{pmatrix} 2 & 3 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

for which a solution is

$$x = \begin{pmatrix} -6 \\ 0 \\ 3 \end{pmatrix}$$

(4) For the first matrix the decomposition reads

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 \\ 0 & 2/3 & 1 & 0 & 0 \\ 0 & 0 & 3/4 & 1 & 0 \\ 0 & 0 & 0 & 4/5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 0 & 5/4 & 1 \\ 0 & 0 & 0 & 0 & 6/5 \end{pmatrix}$$

For the second one, as in exercise 8), we have to multiply by a permutation matrix. Then we do as follows: let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

be the permutation matrix, then

$$PA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = LU$$

(5) The solution to the first part is

$$y = 2x^2 + 3x - 4$$

If we look for a solution of the kind $y = ax^2 + bx + c$, plugging in $(0, -4)$ and $(1, 1)$ we see that

$$\begin{aligned} c &= -4 \\ a + b &= 5, \end{aligned} \tag{0.1}$$

hence the solution is not unique and it is of the form

$$y = \lambda x^2 + (5 - \lambda)x - 4,$$

as λ varies over the real numbers.
For the second part, as hinted, we have to solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

which is equal to the system

$$\begin{pmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{pmatrix} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$$

for which a solution is

$$\begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 5/2 \\ -1 \end{pmatrix}$$

.

(6)

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{becomes} \quad \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\} \quad \text{becomes} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right.$$

$$\left\{ \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \right\} \quad \text{becomes} \quad \left\{ \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \right\}.$$

(7) Here is the MATLAB code:

```
[m, n] = size(A);
Asave = A;
for j = 1:n
  for k = 1:j-1
    mult = (A(:, j)'*A(:, k)) / (A(:, k)'*A(:, k));
    A(:, j) = A(:, j) - mult*A(:, k);
  end
end
for j = 1:n
  if norm(A(:, j)) < sqrt(eps)
    error('Columns of A are linearly dependent.')
  end
  Q(:, j) = A(:, j) / norm(A(:, j));
end
R = Q'*A;
```

(8) As explained in the the second section of Chapter 1.3, in this case, we have to permute the rows of the matrix as the first one begins with 0. We can exchange it with second one and ore new matrix looks like

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and its LU decomposition is

$$L = Id, U = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, we have that if we consider
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
then we have that
$$PA = LU,$$
where we denote by A our original matrix.