18.085, PROBLEM SET 1, SOLUTIONS

(1) a) The solutions to the linear system is

$$u(0) = 0 = u(1),\ u(\frac{1}{4}) = u(\frac{3}{4}) = \frac{3}{32},\ u(\frac{1}{2}) = \frac{1}{8}.$$

b) The values found in a) and those of the analytic solution $u(x) = -\frac{x(x-1)}{2}$ coincide at the points $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

c)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(0) \\ u(\frac{1}{4}) \\ u(\frac{1}{2}) \\ u(\frac{3}{4}) \\ u(1) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{16} \\ 0 \end{pmatrix}$$

d) Let $h = \frac{1}{n}$. Then we want to identify the values of u at the n+1 nodes $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1$.

As the only change form the first part of the problem is in the boundary conditions, we can rewrite the equations for the interior nodes in the same way as before:

$$\begin{cases} \frac{-u(\frac{k-1}{n})+2u(\frac{k}{n})-u(\frac{k+1}{n})}{(\frac{1}{n})^2} & = 1,\ k = 1, 2, 3, \dots, n-2, n-1 \\ u(1) & = 0 \end{cases}$$

As suggested in the hint we should approximate $u'(0)$ with $\frac{u(h)-u(0)}{h}$. Hence, as $u'(0) = 0$, we get that $u(0) = u(\frac{1}{n})$, so that the full linear system to solve the problem looks as follows:

$$\begin{cases} \frac{-u(\frac{k-1}{n})+2u(\frac{k}{n})-u(\frac{k+1}{n})}{(\frac{1}{n})^2} & = 1,\ k = 1, 2, 3, \dots, n-2, n-1 \\ u(1) & = 0 \\ u(0) & = u(\frac{1}{n}) \end{cases}$$

e) The solution to the linear system in d), when n=4 is

$$u(1) = 0,\ u(0) = u(\frac{1}{4}) = \frac{6}{16},\ u(\frac{1}{2}) = \frac{5}{16},\ u(\frac{3}{4}) = \frac{3}{16}.$$

f) The analytic solution computed in class, for the differential equation

$$\begin{cases} -\frac{d^2}{dx^2}u(x) = 1,\ x \in [0, 1] \\ u'(0) = 0 = u(1) \end{cases}$$

is $u(x) = -\frac{x^2-1}{2}$. Hence, in this case

$$u(0) = \frac{1}{2},\ u(\frac{1}{4}) = \frac{15}{32},\ u(\frac{1}{2}) = \frac{1}{8},\ u(\frac{3}{4}) = \frac{7}{32},\ u(1) = 0.$$

So the values obtained in e) and those in f) differ. The reason is given by the error that we introduce when we approximate $u'(0) = 0$ by $\frac{u(h)-u(0)}{h} = 0$.

(2) Code for MAtlab:

```
%define the matrix
A=[0 0 1 0 1;
0 0 0 1 1;
1 0 0 1 0;
1 1 0 0 0;
0 1 1 0 0];
%define the LHS
b=[6 4 5 6 9]';
%x is the vector containing the c values
x=A\b
```

Result:

$$x = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 3 \\ 1 \end{pmatrix}.$$

The column vectors are independent, hence this implies that there is a unique solution $(c_1, c_2, c_3, c_4, c_5)$ that satisfies the above equation, i.e. the vector x above.

(3)

(4)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Null(A)=$\{\lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}\}$, hence dim Null(A)=1 and rk $A = 2$, by the fundamental theorem of algebra. Null(A) is a line described by the equations

$$\begin{cases} 2x + y &= 0 \\ x - z &= 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 6 \\ 1 & 0 & 3 \end{pmatrix}$$

Null(A)=$\{\lambda \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}\}$, hence dim Null(A)=1 and rk $A = 2$, by the fundamental theorem of algebra. Null(A) is a line described by the equations

$$\begin{cases} x + y &= 0 \\ x + \frac{1}{3}z &= 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Null(A)=$\{\lambda \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}\}$, hence dim Null(A)=1 and rk $A = 2$, by the fundamental theorem of algebra. Null(A) is a line described by the equations

$$\begin{cases} x + z & = 0 \\ 2x - y & = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Null(A)= $\{\lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \lambda, \mu \in \mathbb{R}\}$, ihence dim Null(A)=1 and rk $A = 2$, by the fundamental theorem of algebra. Null(A) is a line described by the equations

$$x + y + z = 0$$

(5)

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

This vectors are independent. Hence their span is all of $\mathbb{R}^3$ as they are 3 independent 3-dimensional vectors.

$$\left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

It is immediate to verify that the vectors above satisfy the relation

$$\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ -1 \\ 15 \end{pmatrix} - \frac{31}{3} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The vectors

$$\left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

are nonetheless indepent. Hence

$$\text{span} \left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

This is 2 dimensional, i.e. a plane in $\mathbb{R}^3$, hence it is described by an equation of the type

$$ax + by + cz = 0.$$

Of course, the vectors

$$\left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

satisfy this equation, which imply that

$$\begin{cases} a \cdot 6 + b \cdot 2 + c \cdot 1 = 0 \\ \quad a \cdot 0 + b \cdot 0 + c \cdot 3 \quad = 0 \end{cases}$$

Hence,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$$

for a fixed non zero choice of λ.

(6) a) We already know that $v_1 = 1,\ v_n = 0$. Let's write the conditions imposed by Kirchoff's law at interior nodes:

The total sum of the incoming and the outgoing currents is 0.

This means that, for example at v_2, we have

$$\frac{v_1 - v_2}{R} + \frac{v_3 - v_2}{R} = 0,$$

and the same holds at all interior vertices.
Hence we can write down the following system of equations:

$$\begin{cases} \frac{v_{k-1} - 2v_k + vk+1}{R} & = 0,\ k = 2, 3, \ldots, n-2, n-1 \\ v_1 & = 1 \\ v_n & = 0 \end{cases}$$

When $n = 6$ the above system becomes

$$\begin{cases} \frac{v_1 - 2v_2 + v3}{R} & = 0, \\ \frac{v_2 - 2v_3 + v4}{R} & = 0, \\ \frac{v_3 - 2v_4 + v5}{R} & = 0, \\ \frac{v_4 - 2v_5 + v6}{R} & = 0, \\ v_1 & = 1 \\ v_6 & = 0 \end{cases}$$

In this case the solution is

$$v_1 = 1,\ v_2 = 4/5,\ v_3 = 3/5,\ v_4 = 2/5,\ v_5 = 1/5, v_6 = 0.$$

b)
```
clear all;
n= 10000;
L=sparse([],[],[],n,n,3*n-4);
b=zeros(n,1);

L(1,1)=1;
L(n,n)=1;
```

```
b(1,1)=1;

for i=2:n-1
    L(i,i-1)=1;
    L(i,i)=-2;
    L(i, i+1)=1;
end

%Determine how long it takes to solve Ax=b
tic
v=L\b;
toc

%Print the computed value of node 5000
v5000=sprintf('%0.6f',v(5000));
```

To take into account the time needed to build L, just move the command *tic* at the top of the program. Resolution time is around 0.002 secs.

c)
```
clear all;
n= 10000;
L=zeros(n,n);
b=zeros(n,1);

L(1,1)=1;
L(n,n)=1;
b(1,1)=1;

for i=2:n-1
    L(i,i-1)=1;
    L(i,i)=-2;
    L(i, i+1)=1;
end

%Determine how long it takes to solve Ax=b
tic
v=L\b;
%Print the computed value of node 5000
v5000=sprintf('%0.6f',v(5000));
toc
```

To take into account the time needed to build L, just move the command *tic* at the top of the program.