18.085, PROBLEM SET 7, DUE 8/12 IN CLASS.

Question 1. (40 pts.) Fourier Series

a) f is an even function and so its Fourier serie will be a eerie of cosines,

$$f(x) = \sum_{\natural \ni n\ ,n \text{ odd}} (-1)^{\frac{n-1}{2}} \frac{\cos nx}{n}.$$

b) The form of the Fourier eerie of F tells us that $F_2 = F_1$ and $F_4 = F_3$. Here are the graphs of F_3 and F_5.

c) (1) The Fourier serie of $f(x) = x$ is just a sines serie as the function is odd:

$$f(x) = \sum_{n \in \mathbb{N}} (-1)^{n+1} \frac{2 \sin kx}{n}$$

(2) The Fourier serie of $f(x) = |\sin x|$ is a cosine serie as the function is even:

$$f(x) = \frac{2}{\pi}(1 - \sum_{n=2,\ n \text{ even}} \frac{\cos(nx)}{(n-1)(n+1)})$$

(3) The complex form of the Fourier serie of $f(x) = \exp x$ is

$$f(x) = \sum_{n \in \mathbb{Z}} \frac{\sinh((1-ni)\pi)}{\pi(1-ni)} e^{inx}$$

d) We showed that the energy of a function $g(x)$, whose Fourier serie is $g(x) = \sum_{k \in \natural} b_k \sin kx + c_k \cos kx$ is

$$\int_0^{2\pi} |g(x)|^2 dx = 2\pi c_0^2 + \pi \sum_{k=1}^{\infty} (b_k^2 + c_k^2)$$

The Fourier serie of f is

$$f(x) = \frac{4}{\pi} \sum_{k=1,\ k \text{ odd}}^{\infty} \frac{\sin kx}{k},$$

hence the above equality becomes

$$\int_0^{2\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} 1 dx = 2\pi = \pi(\frac{16}{\pi^2}) \sum_{k=1,\ k \text{ odd}}^{\infty} \frac{1}{k^2}$$

which we can rewrite as

$$\frac{\pi^2}{8} = \sum_{n \in \mathbb{N},\ n \text{ odd}} \frac{1}{n^2}.$$

Question 2. (30 pts.) Unitary matrices.

a)

$$v \cdot Uw = v^T \overline{Uw} = v^T \overline{U}\,\overline{w} = (\overline{U} v)^T \overline{w} = U^* v \cdot w$$

b) The answer is yes for both matrices. If U is unitary then

$$U\overline{U}^T = UU^* = Id.$$

If we transpose both sides of the equation, we get

$$\overline{U}^{T^T} U^T = Id.$$

$\overline{U}^{T^T} = \overline{U}$, hence

$$\overline{U}^{T^T} U^T = \overline{U} U^T = (U^T)^* U^T = Id,$$

which implies that U^T is unitary. If $(U^T)^* = \overline{U}$, hence the claim that if a matrix is unitary then the $*$ matrix is also unitary terminates the proof.

c) The answer is yes. If U, V are unitary then

$$UU^* = Id = VV^*.$$

What is the $(UV)^*$?

$$(UV)^* = (\overline{UV})^T = (\overline{U}\,\overline{V})^T = \overline{V}^T \overline{U}^T = V^* U^*.$$

Then

$$UV(UV)^* = UVV^*U^* = UIdU^* = UU^* = Id.$$

is UV unitary?

d) For any $v \in \mathbb{C}^N$,

$$Uv \cdot Uv = (Uv)^T \overline{Uv} = v^T U^T \overline{U}\,\overline{v} = v^T U^T (U^T)^* \overline{v}.$$

We saw in part b), that if U is unitary, then also U^T is unitary. Then the last part of the equality becomes

$$v^T U^T (U^T)^* \overline{v} = v^T \overline{v} = v \cdot v.$$