18.085, PROBLEM SET 6 SOLUTIONS

Question 1. Newton's method.

We will be using Newton's method to find the roots of polynomials. We will be using the Matlab file on the website.

a) Take $g(u) = u$, then $\frac{dg}{du}(u) = 1$. We change line 16 of the code so it runs Newton's method:

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u1=u0-g(u0)/dg(u0);
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The starting guess is $u_0 = 1/2$. We run the code with that starting guess, and three other starting guesses of our choice. In every case, the method converges in exactly 1 step. This is because our function is a straight line: Newton's method follows that line until it crosses the x-axis in order to find the next guess, but it happens that for a function which is a line, that next guess is the exact solution.

Now, a slightly harder problem: $g(u) = \frac{1}{3}u^3 - \frac{3}{2}u^2 + 2u$.

b) The derivative is $\frac{dg}{du}(u) = u^2 - 3u + 2$. See solution code for how to put this in the code. Back to a starting guess $u_0 = 1/2$. We run the code and see it converges to approximate root $u^* = -1.6324 * 10^{-10} \approx 0$ with $|g(u^*)| = -3.2649e - 10 \approx 0$.

c) The derivative is $\frac{dg}{du} = u^2 - 3u + 2 = (u-2)(u-1)$, hence the roots are at 1 and 2. Since Newton's method needs to divide by the derivative at each guess, and we have starting guesses where the derivative is 0, the method will not converge after 100 iterations and return infinity for the last guess (or NaN in Matlab, we won't discuss the subtleties of Inf vs NaN for Matlab).

d) We plot the polynomial g as a function of u in figure 1. We see there is only one root, and the guess $u_0 = 3/2$ is between the relative minimum and maximum. What happens is that Newton's method struggles here: it wants to go right to go towards 0, but the function at the min $u = 2$ is positive. So then it goes left, but again stays stuck in the "bowl" created by the relative min. It takes a number of iterations, and going left and right around $u = 2$, before Newton's method can escape the "bowl" and finally converge.

Relative minima and maxima can be treacherous for Newton's method, both because they exhibit a 0 derivative, and also because the method can get "stuck" there for a number of iterations.

Question 2. Finite elements.

We will solve $-u'' = x$ for $u = u(x)$ in $[0, 1]$ with boundary conditions $u(0) = u(1) = 0$ using the Finite Elements Method. Let $h = 1/3$.

a) We will need 2 hat functions ϕ_1 and ϕ_2 since there are 2 free nodes, at $x = 1/3$ and $2/3$. Nodes at $x = 0$ and 1 are fixed at 0. See the hat functions on figure 2 and their derivatives on figure 3.

b) Assume $\phi_1 = V_1$, $\phi_2 = V_2$. Recalling that $h = 1/3$ and splitting the integral into elements: $K_{12} = \frac{1}{3} \cdot 3 \cdot 0 + \frac{1}{3} \cdot -3 \cdot 3 + \frac{1}{3} \cdot 0 \cdot -3 = -3 = K_{21}$.

c) $K_{11} = \frac{1}{3} \cdot 3^2 + \frac{1}{3} \cdot (-3)^2 + \frac{1}{3} \cdot 0^2 = 6 = K_{22}$.

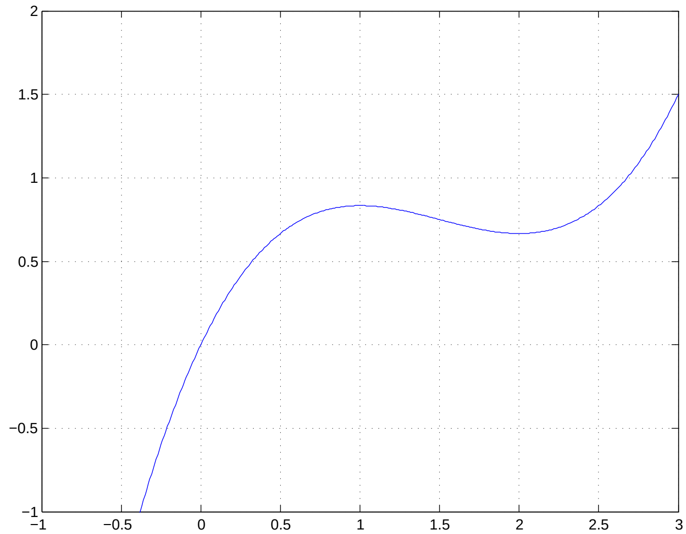

FIGURE 1. Plot of g for 1.b), c), d).

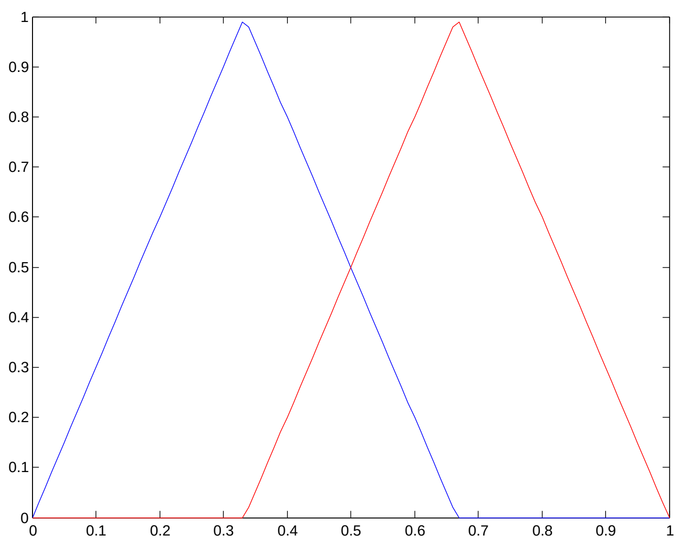

FIGURE 2. Hat functions.

d) Break up the $[0,1]$ interval into elements, and use the midpoint rule on each element separately. We have $f(x) = x$. $F_1 = \frac{1}{3} \cdot \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{6} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{5}{6} \cdot 0 = \frac{1}{9}$. $F_2 = \frac{1}{3} \cdot \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot \frac{3}{6} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{5}{6} \cdot \frac{1}{2} = \frac{2}{9}$.

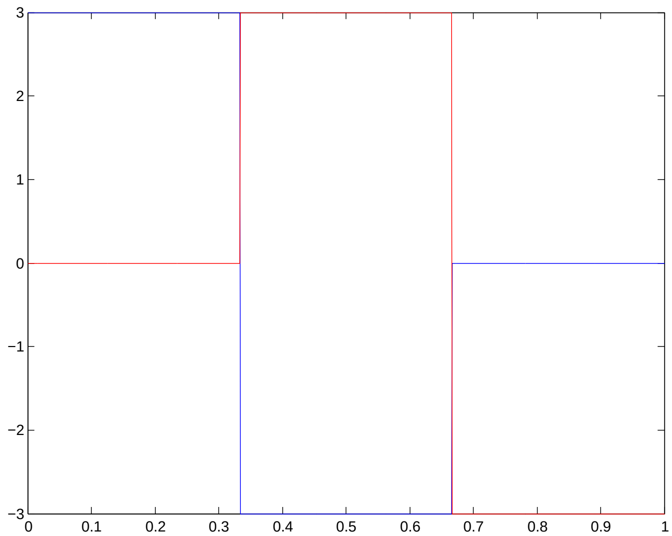

FIGURE 3. Derivatives of hats.

e) Finally, solve the system $K\vec{U} = \vec{F}$ for $\vec{U}$:

$$K\vec{U} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 2/9 \end{pmatrix}$$

or

$$K^{-1}\vec{F} = \frac{1}{36-9} \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1/9 \\ 2/9 \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \frac{1}{27} \begin{pmatrix} 4/3 \\ 5/3 \end{pmatrix}.$$

f) The approximate solution is $U(x) = U_1\phi_1(x) + U_2\phi_2(x)$. We know it is 0 at $x = 0$ and $x = 1$, it is $4/81$ at $x = 1/3$ and $5/81$ and $x = 2/3$, and linear in-between. See figure 4.

Question 3. (20 pts.) Potentials and stream functions.

a) The potential of v is

$$u = \frac{x^2y^2 + x^2z^2 + y^2z^2}{2} + cost.$$

b) If a potential u existed, then we could recover it, integrating by parts as follows:

$$u(x, y) = \int v_x dx + F(y) = \int v_y dy + G(x)$$

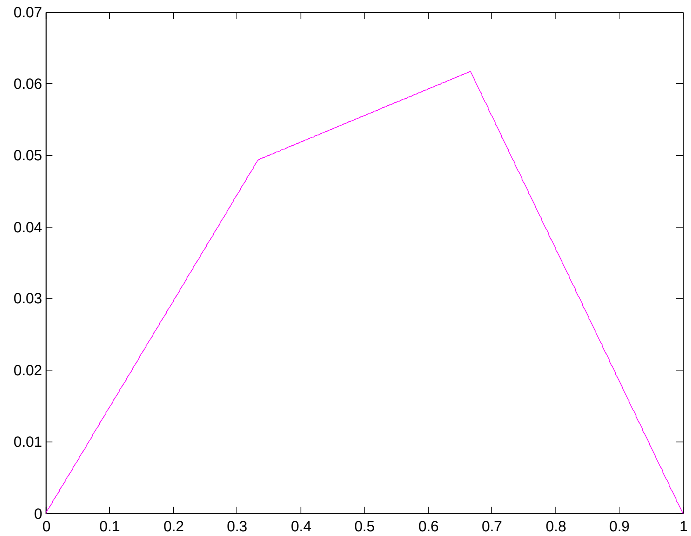

FIGURE 4. Solution.

where F, G are functions of one variable.
Now,

$$u(x,y) = \int v_x dx + F(y) = 2xy + F(y)$$

$$u(x,y) = \int v_y dy + G(x) = -2\lambda xy + G(x), \text{ hence,}$$

$$2xy + F(y) = -2\lambda xy + G(x).$$

This forces $\lambda = -1$ in order to obtain a potential u, in which case $u = 2xy + cost$.
When $\lambda \neq -1$ then we cannot find a potential, as $curl\ v \neq 0$.

c) As $\frac{\partial v_1}{\partial y} = 2x$, $\frac{\partial v_2}{\partial x} = 2x$, then $curl\ v = 0$, hence v is conservative and the potential is given by $u(x,y) = x^2 y - \frac{y^3}{3}$.
Since $div\ v = 0$, as we discussed in class, v also admits a stream function s.
Such stream function will be of the form $s(x,y) = \frac{x^3}{3} - xy^2$.

Question 4. (20 pts.) Laplace's equation.

a) It is immediate to compute that

$$u_{xx} = -k^2 \frac{\sin(\pi k x)\sinh(\pi k y)}{\sinh(\pi k)}$$

$$u_{yy} = k^2 \frac{\sin(\pi k x)\sinh(\pi k y)}{\sinh(\pi k)}, \text{ hence,}$$

$$\Delta u = 0.$$

c) This follows from parts a) and b) and the fact that we can take derivatives of the serie term by term.

d) The solution is $v(x,y) = \sum_k \frac{\sin(\pi k x)\sinh(\pi k(1-y))}{\sinh(\pi k)}$