18.085, PROBLEM SET 5, SOLUTIONS

Question 1. (40 pts.) Standing waves revisited.
We solve

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t), \qquad x \in [0,1], \qquad u(0) = u(1) = 0. \tag{0.1}$$

with $u(x,t) = \sin(k2\pi x)\cos(k2\pi t)$ (k some real number), with initial conditions $u(x,0) = \sin(k2\pi x)$ and $\frac{\partial}{\partial t}u(x,0) = 0$.

a) Verify our guess for u satisfies both initial conditions:

$$u(x,t=0) = \sin(k2\pi x)\cos(k2\pi t)|_{t=0} = \sin(k2\pi x)$$

works. Also,

$$\frac{\partial}{\partial t}u(x,t=0) = (k2\pi)\sin(k2\pi x)\sin(k2\pi t)|_{t=0} = 0,$$

which also works. There is no restriction on the possible values of k.

b) Verify our guess for u satisfies both boundary conditions:

$$u(x=0,t) = \sin(k2\pi 0)\cos(k2\pi t) = 0$$

works. Also, $u(x=1,t) = \sin(k2\pi 1)\cos(k2\pi t) = 0$ implies we need $\sin(k2\pi) = 0$, which implies $k2\pi$ needs to be an integer multiple of π, so $2k$ needs to be an integer. Hence we get a restriction on the possible values of k:

$$k = \cdots - 1.5, -1, -0.5, 0, 0.5, 1, 1.5, \ldots.$$

c) Verify our guess for u satisfies the wave equation (0.1). We got earlier

$$\frac{\partial}{\partial t}u(x,t) = (k2\pi)\sin(k2\pi x)\sin(k2\pi t)$$

and so

$$\frac{\partial^2}{\partial t^2}u(x,t) = (-1)(k2\pi)^2\sin(k2\pi x)\cos(k2\pi t) = (-1)(k2\pi)^2u(x,t).$$

Also,

$$\frac{\partial}{\partial x}u(x,t) = (k2\pi)\cos(k2\pi x)\cos(k2\pi t)$$

and

$$\frac{\partial^2}{\partial x^2}u(x,t) = -(k2\pi)^2\sin(k2\pi x)\cos(k2\pi t) = -(k2\pi)^2u(x,t).$$

Hence it is clear that the wave equation $\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$ is satisfied. We do not get an additional restriction on the possible values of k.

d) Based on answers to (a), (b), (c), we guess $u(x,t) = \cos(k2\pi x)\cos(k2\pi t)$ for the solution of (0.1) if we changed the boundary conditions to be free-free ends, that is, $\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(1,t) = 0$. You can check for yourself that the initial conditions are still satisfied, and that the wave equation itself is also still satisfied. Now, boundary conditions: $\frac{\partial}{\partial x}u(x,t) = -\sin(k2\pi x)\cos(k2\pi t)$, so that $\frac{\partial}{\partial x}u(x=0,t) = 0$ is satisfied, and to get also $\frac{\partial}{\partial x}u(x=1,t)=0$, we need $\sin(k2\pi 1)=0$ or again k can be half-integers: $k=\cdots-1.5,-1,-0.5,0,0.5,1,1.5,\ldots$. This is the appropriate restriction on k.

Others tried a phase change in x, that is $u(x,t)=\sin(k2\pi x-\phi)\cos(k2\pi t)$, which would give the same answer (up to a minus sign) once the value of $\phi=\pm\pi/2$ is found.

So in problem set 4, we could have tried half-integers too for standing waves! And cosines with free-free ends.

Question 2. (30 pts.) Resonance and practice with (complex) exponentials.
Consider :

$$mu''+bu'+ku=f. \tag{0.2}$$

Again, we guess that $u=e^{rt}$ where r is some complex number to be found (we can't expect anymore that $r=i\omega=i\sqrt{k/m}$, the natural frequency of the *un-damped* system).

a) Let $f=0$. Verify our guess for u satisfies (0.2): $u'=re^{rt}$ and $u''=r^2e^{rt}$ so $mu''+bu'+ku=(mr^2+br+k)e^{rt}=0$. Since e^{rt} is never 0, we solve the quadratic in r to get 2 possible values of r: $r=\frac{-b\pm\sqrt{(b^2-4mk)}}{2m}$. This is the restriction on the possible values of r.
b) Put $b=0$ (no damping) in your answer for r from (a) and verify that this corresponds to what we expect for the un-damped system: now the possible values for r are $r=\frac{\pm\sqrt{(-4mk)}}{2m}=\pm i\sqrt{k/m}$, as expected.
c) Let $b=0$ again, and now let $f(t)=e^{rt}$, where r is one of the (perhaps multiple) possible values of r you got in (b). Clearly the guess $u=e^{rt}$ does not satisfy (0.2) with $b=0$ and $f(t)=e^{rt}$, since we solved the equation $mu''+bu'+ku=0$ with it, and now we want a non-zero right-hand side!
d) Same as in (c), but now try the guess $u=Ate^{rt}$, where A is the amplitude, possibly a complex number: $u'=A(1+tr)e^{rt}$ and $u''=A(r+r+tr^2)e^{rt}$, so $mu''+ku=Ae^{rt}\left(2mr+mtr^2+kt\right)=f=e^{rt}$, but $r^2=-k/m$ from (c), so that in fact we have the equality $Ae^{rt}\left(2mr+mt(-k/m)+kt\right)=e^{rt}$ or $A2mr=1$, or we get the following restriction on A: $A=1/2mr$, with $r=\pm i\sqrt{(k/m)}$. That is, $A=\pm\frac{i}{2\sqrt{mk}}$.

This proves that, if you force a system such as (0.2) at its natural frequency, the amplitude of your solution will grow in time (but will still oscillate). This is resonance.

Question 3. (30 pts.) Structures, or how to build a stable treehouse.
We will look at the treehouses on figures 2.30, 2.31 of the book.

a) We have the following matrix for the truss (changing the sign of all entries in one row, for any row, is also a valid answer):

$$A = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 & -\cos\theta_1 & -\sin\theta_1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and clearly the mechanism $u_2^V = u_5^V = \Delta$ is in the null space of matrix A: let $u = (0,0,0,\Delta,0,0,0,0,0,\Delta)^T$, then clearly $Au = 0$. The treehouse is unsafe.

b) Add a new bar in the treehouse as in Figure 2.31:

$$A = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 & -\cos\theta_1 & -\sin\theta_1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\phi & \sin\phi \end{pmatrix},$$

and a basis for the null space of the new A is, as suggested in the book, $u = (\Delta\sin\phi, 0, \Delta\sin\phi, -\Delta\cos\phi, \Delta\sin\phi, 0, \Delta\sin\phi, 0, \Delta\sin\phi, -\Delta\cos\phi)^T$. You can verify this is in the nullspace: $Au = 0$. To find this, reason this way: node 5 will move on a circle around node 7. To first approximation, that means it will move at a right angle to bar 9, so with angle ϕ with the vertical. Then its displacement is $\Delta\sin\phi$ horizontally, and $-\Delta\cos\phi$ vertically. This means that nodes 1, 4 and 3 will also have a horizontal displacement of $\Delta\sin\phi$. Is there another independent vector in the nullspace? Our intuition would say no, because A is now 9 by 10. But we could have dependent rows. To convince ourselves the rows are independent (hence the rank of A is 9, hence its nullspace has dimension 10-9=1), we can reason this way. First, notice row 7 forces $u_6 = u_3^V = 0$ and row 8 forces $u_8 = u_4^V = 0$ so we may remove rows 7, 8 from A, along with columns 6 and 8. We get a smaller matrix, and it is easier to see then that the rows are independent:

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 & -\cos\theta_1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\phi & \sin\phi \end{pmatrix}.$$

c) Since A is 9 by 10, with rank 9, we only need to add one bar, so one row which is independent from the others. We could add a bar, say, between nodes 6 and 4, and get the new A:

$$A = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 & -\cos\theta_1 & -\sin\theta_1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\theta_2 & \sin\theta_2 & 0 & 0 \end{pmatrix},$$

and you can convince yourself that this new row is *not* a combination of the other rows. Then the columns are independent too. You could also try a few values of the angles θ_1, θ_2 and ϕ, put this A in Matlab and see it is invertible.