18.085, PROBLEM SET 7, DUE 8/12 IN CLASS.

Question 1. (40 pts.) Fourier Series

a) Consider the function

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, \pi/2] \\ -1 & \text{for } x \in (\pi/2, 3\pi/2) \\ 1 & \text{for } x \in [3\pi/2, 2\pi] \end{cases}$$

Is it an even or an odd function or neither of them? What does that tell you about its Fourier series? Compute the Fourier series of f,

$$f(x) = \sum_{k \in \mathbb{N}} (b_k \sin(kx) + c_k \cos(kx)). \tag{0.1}$$

b) Let us call F_N the truncation at index N of the Fourier series of the function $f(x)$ just defined,

$$F_N(x) = \sum_{k=0}^{N} (b_k \sin(kx) + c_k \cos(kx)),$$

where the coefficients b_k, c_k are those you computed in (0.1). Sketch the behavior of F_N on $[0, 2\pi]$ (or, if you prefer, on $[-\pi, \pi]$) for $N = 1, 2, 3, 4, 5$.

c) Compute the Fourier series of the following functions on $[-\pi, \pi]$:
  (1) $f(x) = x$;
  (2) $f(x) = |\sin x|$;
  (3) $f(x) = \exp x$, using the complex form of the Fourier transform that we saw in class on Monday.

d) Using the Fouries series we computed in class for the function

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, \pi] \\ -1 & \text{for } x \in (\pi, 2\pi) \end{cases}$$

compute its energy i.e.

$$\int_0^{2\pi} |f(x)|^2 dx$$

to derive the following interesting equality

$$\frac{\pi^2}{8} = \sum_{n \in \mathbb{N},\, n \text{ odd}} \frac{1}{n^2}.$$

Question 2. (30 pts.) Unitary matrices.

Recall that an $N \times N$ matrix A, with complex coefficients, is said to be unitary if $AA^* = Id$, where $A^* = \overline{A^t}$, i.e. $A^{-1} = A^*$. Let me remind you how we defined the dot product among vectors $v, w \in \mathbb{C}^N$.

If $v = (v_1, v_2, \dots, v_N)^t, w = (w_1, w_2, \dots, w_N)^t$, then

$$v \cdot w = \sum_{i=1}^{N} v_i \overline{w}_i = (v_1, v_2, \dots, v_N) \begin{pmatrix} \overline{w_1} \\ \overline{w_2} \\ \vdots \\ \overline{w_N} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}^t \overline{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}},$$

where

$$\overline{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}} = \begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \overline{a_N} \end{pmatrix}$$

for any vector $(a_1, a_2, \dots, a_N) \in \mathbb{C}^N$.

a) Prove that for a unitary matrix U, the following equality holds

$$U^* v \cdot w = v \cdot Uw$$

for any two vectors $v, w \in \mathbb{C}^N$, where $v \cdot w$ is the dot product on $\mathbb{C}^N$ we defined in class.

b) If U is unitary then also U^* is unitary. Are $\overline{U}, U^t$ unitary?

c) If U, V are unitary, is UV unitary?

d) If U is unitary, then

$$Uv \cdot Uv = v \cdot v,$$

for any $v \in \mathbb{C}^N$.