18.085, PROBLEM SET 6, DEFINITIVE VERSION, DUE 8/6 BY 10AM, E18-401 WST
(LOOK FOR TIANKAI LIU'S DESK IN THE BIG ROOM OF THE GRADUATE STUDENTS)

Question 1. (20 pts.) Newton's method.
We will be using Newton's method to find the roots of polynomials. We will be using the Matlab file on the website.

a) Take $g(u) = u$, then $\frac{dg}{du}(u) = 1$. Change line 16 of the code so it runs Newton's method. The starting guess is $u_0 = 1/2$. Run the code with that starting guess, and three other starting guesses fo your choice, and report what happens. Explain why we expect this (we talked about this in class for the special case of a line).

Now, a slightly harder problem: let g be the third degree polynomial: $g(u) = \frac{1}{3}u^3 - \frac{3}{2}u^2 + 2u$.

b) Find the derivative of $g(u)$ and change lines 7 and 8 of the code accordingly. Back to a starting guess $u_0 = 1/2$. Notice how we have a tolerance of $tol = 10^{-6}$ and a maximum number of iterations $stp = 100$. Run the code and say whether it converges, and if so what is the approximate root u^* so that $g(u^*) \approx 0$?

c) Find the root(s) of the derivative $\frac{dg}{du}$. Try those roots as starting guesses for the algorithm. What happens, and why?

d) Plot the polynomial g as a function of u. This will help you to answer the following. Try $u_0 = 3/2$ as a starting guess. Uncomment the part of line 16 that will display the next guesses, and run the code. Report what happens.

Question 2. (40 pts.) Finite elements.
We will solve $-u'' = x$ for $u = u(x)$ in $[0,1]$ with boundary conditions $u(0) = u(1) = 0$ using the Finite Elements Method. Let $h = 1/3$.

a) How many hat functions ϕ will you need? Hint: think of the boundary conditions, and how many "free" nodes there are. Draw the hat functions together on one plot, and their derivatives together on a different plot.

b) Let's find the matrix K first. We assume $\phi_1 = V_1$, $\phi_2 = V_2$, ... Hence $K_{12} = K_{21}$. Find the entries $K_{12} = K_{21}$ by exact integration (no mid-point rule, exact integration is easy here). Don't forget that sometimes it is easier to split the intergal into as many parts as there are elements, and add the results together.

c) Find K_{11} and K_{22}. You should find that they are equal (this is because of the symmetry, since we use the same boundary condition on each side of the $[0,1]$ interval.)

d) Now we find the vector $\vec{F}$. Use the mid-point rule. Again, break up the $[0,1]$ interval into elements, and use the midpoint rule on each element separately.

e) Finally, solve the system $K\vec{U} = \vec{F}$ for $\vec{U}$.

f) What is the approximate solution $U(x) = U_1\phi_1(x) + U_2\phi_2(x)$? Draw it in $[0,1]$

Question 3. (20 pts.) Potentials and stream functions.

a) Verify that $curl\ v = 0$, when $v = (x(y^2+z^2), y(z^2+x^2), z(y^2+x^2))$. This v must be the gradient of some $u(x,y,z)$. What is u?

b) Using the method shown in class (i.e. trying to integrate the partial derivatives), show that it is *not* possible to find a potential for the vector field

$$v = (2y, -2\lambda x), \text{for } \lambda \text{ a fixed constant,}$$

unless $\lambda = -1$. Why can't you find a potential in the other cases?

c) Consider the vector field

$$v = (2xy, x^2 - y^2).$$

Is it conservative? If so, find a potential u. Does is admit a stream function s? If it admits a potential u and/or a stream function s, try to sketch, in the first quadrant the level sets of u and/or s, i.e. the curves given by $\{u(x, y) = c\}$ (the so-called equipotentials) and $\{s(x, y) = d\}$ (the so-called streamlines), when c and d vary in the real. What do you notice in this case?

Question 4. (20 pts.) Laplace's equation.

In the following exercise we are going to construct some solutions to Laplace's equation. Recall that $\sinh z = \frac{e^z - e^{-z}}{2}$.

a) Verify that

$$u_k(x, y) = \frac{\sin(\pi k x) \sinh(\pi k y)}{\sinh(\pi k)}$$

solves Laplace's equation, for $k = 1, 2, 3, \ldots$.

b) What are the boundary values of u_k on the unit square?

c) Suppose that $u_0 = \sum_k b_k \sin(\pi k x)$ along the top edge $y = 1$ of the unit square and $u_0 = 0$ on the other three edges. by linearity, $u(x, y) = \sum_k b_k u_k(x, y)$ solves Laplace's equation with those boundary values. Verify this.

d) What is the solution if $u_0 = \sum_k c_k \sin(\pi k x)$ along the bottom edge $y = 0$ and zero on the other three edges?