18.085, PROBLEM SET 5, DUE 7/22 IN CLASS

No (or almost no) Matlab this week!

Question 1. (40 pts.) Standing waves revisited.
Consider the wave equation for the height u of a string,

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial}{\partial x}\left(c(x)\frac{\partial}{\partial x}u(x,t)\right) + f,$$

say in the unit x interval, with fixed ends and constant stiffness 1, and no forcing, that is, we solve

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t), \qquad x \in [0,1], \qquad x(0) = x(1) = 0. \tag{0.1}$$

Because of what we tried in the previous problem set, question (c), we expect $u(x,t) = \sin(k2\pi x)\cos(k2\pi t)$ to be a solution (k some real number), if our initial conditions are $u(x,0) = \sin(k2\pi x)$ and $\frac{\partial}{\partial t}u(x,0) = 0$.

a) Verify our guess for u satisfies both initial conditions. Do you get a restriction on the possible values of k?
b) Verify our guess for u satisfies both boundary conditions. Do you get a restriction on the possible values of k?
c) Verify our guess for u satisfies the wave equation (0.1). Do you get a restriction on the possible values of k?
d) Based on your answers to (a), (b), (c), what would you suggest as a guess for the solution of (0.1) if we changed the boundary conditions to be free-free ends, that is, $\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(1,t) = 0$? What would be the appropriate restriction on k?

Feel free to go back to problem set 4 and try your answers, see if it seems to be working!

Question 2. (30 pts.) Resonance and practice with (complex) exponentials.
In this question, $i = \sqrt{-1}$, the unit imaginary number. Recall the equation for one mass and one spring, where u is the position of the mass away from equilibrium, is $mu'' + ku = 0$ (no forcing), where the prime $'$ means a derivative in time, and $u = u(t)$ is a function of time only (no space). Now we consider a slightly more general equation :

$$mu'' + bu' + ku = f. \tag{0.2}$$

The term with b means we have added damping to the system. As usual, f is some external forcing. You can probably guess what the solution would "look" like, but let's compute it! Again, we guess that $u = e^{rt}$ where r is some real number to be found (we can't expect anymore that $r = i\omega = i\sqrt{k/m}$, the natural frequency of the *un-damped* system). Notice that we have not defined any initial conditions for now, and we don't have boundary conditions since there is no x here.

a) Let $f = 0$. Verify our guess for u satisfies (0.2). Do you get a restriction on the possible values of r?
b) Put $b = 0$ (no damping) in your answer for r from (a) and verify that this corresponds to what we expect for the un-damped system.

c) Let $b = 0$ again, and now let $f(t) = e^{rt}$, where r is one of the (perhaps multiple) possible values of r you got in (b). Does the guess $u = e^{rt}$ still satisfy (0.2) with $b = 0$ and $f(t) = e^{rt}$?

d) Same as in (c), but now try the guess $u = Ate^{rt}$, where A is the amplitude, possibly a complex number. Does this guess satisfy (0.2) with $b = 0$ and $f(t) = e^{rt}$? (Again, r as in (b).) Do you get a restriction on A?

This proves that, if you force a system such as (0.2) at its natural frequency, the amplitude of your solution will grow in time (but will still oscillate). This is resonance.

Question 3. (30 pts.) Structures, or how to build a stable treehouse.
We will look at the treehouses on the bottom of pages 192-193 (figures 2.30, 2.31) of the book.

a) Build the matrix A corresponding to the first treehouse, on page 192, with nodes 6 and 7 fixed (hence don't include them in your matrix), so the truss shown on the right of figure 2.30. Show that the mechanism they describe ($u_2^V = u_5^V = \Delta$) is in the null space of matrix A, that is, it is a possible mechanism (and the treehouse is unsafe).

b) Add a new bar in the treehouse as in Figure 2.31, modify A accordingly, and show that, again, the treehouse is unsafe. In particular, find a basis for the null space of the new A.

c) Take the treehouse in (b), and make it safe, that is, add as *few* bars as possible to make it safe. Do *not* add new nodes. Prove your new treehouse is safe by showing that the columns of the corresponding A are independent (you might want to use Matlab for that). Do *not* give a physical explanation - it's easy to make mistakes in this way, but the math is always clear.

Note: lengths of bars don't matter! Assume bars 3, 4, 7, 8 are vertical, bars 2, 5, 6 are horizontal, and bars 1 and 5 make an angle θ_1, as shown on the left of figure 2.30. And assume bar 9, when you add it in (b), makes an angle of ϕ with the horizontal as shown on the left of figure 2.31. When you add a bar in (c), make a drawing of where you will put it and what angle it will be at (you can use a variable for that too).