18.085, PROBLEM SET 2, DUE 6/24 IN CLASS

Definitive version

1) (10 pts.) Prove whether or not the following matrices are invertible. If they are, then write down their inverse matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 110 & 0 \\ 0 & 0 & 11110 \end{pmatrix}.$$

2) (10 pts.) By means of trial and error, write down 2x2 matrices for each of the following properties:

a) $AB \neq BA$
b) $C^2 = 0$ and none of entries of C are 0.
c) $D^2 = -I$ and the entries of D are all real numbers.
d) $E^T = E^{-1}$.

3) (10 pts.) By hand, factor the following matrices into their LU decomposition

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix}$$

4) (10pts) Let L and U be respectively the following two matrices

$$L = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solve the linear system $LUx = f$ as we explained in class, by splitting it into 2 different linear system and doing backward substitution, when

$$f = \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$$

5) (10 pts.) Show that there is a unique parabola of the form $y = 2x^2 + bx + c$ going through the points

$$(0, -4), (1, 1).$$

Does uniqueness still hold if we try instead to fit a parabola of the form $y = ax^2 + bx + c$?

Show that there is no parabola of the form $y = ax^2 + bx + c$ that contains the points.

$$(0, 0),\ (1, 1),\ (2, 1),\ (1, 2).$$

Try to find the best quadratic fit for these points.

Hint: use the method explained in class, by solving the linear system

$$A^t A u = A^t b.$$

Do not use Matlab for this exercise!

6) (10 pts.) Using the Gram-Schmidt algorithm, find the orthonormal basis associated to each of the following bases of $\mathbb{R}^2$ or $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \right\}.$$

7) (20 pts.) Write a Matlab code that given an $n \times n$ matrix A runs the Gram Schmidt process on the columns of A and actually tells whether or not the vector are dependent. Finally, compare the output that you get with the QR decomposition of A. *Hint* : to compute the QR decomposition of A, you just need to use the command

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[Q, R] = qr(A)
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Also remember from what we discuss in class, that if you a vector v_k is dependent from the previous ones then once you run the Gram Schmidt algorithm on that vector you obtain the 0 vector. Hence, you won't be able to normalize such vector, as you would be dividing by 0. This means, you should try to introduce a checking structure, after running Gram Schmidt, and before normalizing.

Apply your algorithm to the following matrices

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 109 & 0 & 1 \end{pmatrix}.$$

8) (15 pts.) Compute the LU decomposition for the following matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hint: you may want to take a look at the first two paragraphs of the chapter 1.3 in the book.