18.085, PROBLEM SET 1, DUE 6/17 IN CLASS

Notice: definitive version.

Please state clearly what exercise (and/or which portion of exercise) are you solving every time you start a solution. Try also to write as clear as possible, as we will have to go through all your solutions.

1) (25 pts.) Complete the example explained in class.

Here's the setting. We want to find an approximate solution, using finite differences, for the differential equation (with boundary conditions)

$$\begin{cases} -\frac{d^2}{dx^2}u(x) = 1, \ x \in [0,1] \\ u(0) = 0 = u(1). \end{cases} \tag{0.1}$$

As we explained in class, we saw that in the case where the mesh is $h = \frac{1}{4}$, in order to find the values of u that we want to interpolate, we have to solve the following linear system:

$$\begin{cases} u(0) = 0 = u(1) \\ \frac{-u(0)+2u(\frac{1}{4})-u(\frac{1}{2})}{(\frac{1}{4})^2} = 1 \\ \frac{-u(\frac{1}{4})+2u(\frac{1}{2})-u(\frac{3}{4})}{(\frac{1}{4})^2} = 1 \\ \frac{-u(\frac{1}{2})+2u(\frac{3}{4})-u(1)}{(\frac{1}{4})^2} = 1 \end{cases}$$

a) (4 pts.) Solve this linear system.

b) (4 pts.) Do the values that you found for u at the nodes, by solving the above linear system, coincide with the values of the solution that we computed in class for the differential equation in (0.1)?

c) (2 pts.) Write down the above linear system in the form $Ax = b$.

d) (7 pts.) Consider now the second differential equation (with new boundary conditions!) that we discussed in class

$$\begin{cases} -\frac{d^2}{dx^2}u(x) = 1, \ x \in [0,1] \\ u'(0) = 0 = u(1) \end{cases} \tag{0.2}$$

How can we set up the discretization problem? (Hint: Notice that the only thing changing is one of the boundary conditions. In (0.1) we had $u(0) = 0$. Here we have $u'(0) = 0$. How do you realize this condition in our setting? You might try to remember that from the discussion in class we had that by using Taylor polynomials, we could make the following approximation

$$u(x+h) = u(x) + u'(x)h + (\text{terms of order at least 2 in h}),$$

hence you could approximate $u'(x)$ by $\frac{u(x+h)-u(x)}{h}$.)

e) (3 pts.) Solve the linear system in the case $h = \frac{1}{4}$.

f) (5 pts.) Do the values that you found in e) coincide with values of the actual solutions that we saw in class at each nodes? Can you explain why?

2) (10 pts.) Find c_1, c_2, c_3, c_4, c_5 such that

$$\begin{pmatrix} 6 \\ 4 \\ 5 \\ 6 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_5 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Put the above problem in the form of a linear system $Ax = b$ and show how to obtain the same solution in this context. Let Matlab do the computation.
Do the column vectors on the right hand side above form a system of independent vectors? In how many different ways can we choose the c_i's so that they satisfy the above equality?

3) (10 pts.) Find and sketch the null space of the following matrices:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 6 \\ 1 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find the rank of these matrices. (Do not use Matlab for this exercise)

4) (10 pts.) Describe the span of the following sets of vectors.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

For each set, find a maximal system of independent vectors. What is then the dimension of their span? (When I say describe the span, please, provide equations for the vector subspace given by the span. Of course if the span is all of $\mathbb{R}^3$, there is no need for equations. Do not use Matlab for this exercise.)

5) (25 pts) Consider a line of n nodes, each connected to its neighbors by a resistor of resistance R. At the first node, v_1, potential is set to 1. At the n-th node, potential is set to 0.

a) (8 pts.) Write down n equations relating $v_1, v_2, \ldots, v_n$, the voltage at each of the nodes. For $n = 6$, write out by hand the equations in the form $Lv = b$.

b) (12 pts.) Write a Matlab program, that for arbitrary n, forms L (as a *sparse* matrix) and b and solves for v. Solve $Lv = b$ in the case $n = 10^4$. What is the computed value of v_{5000}? Provide 6 digits. How long does it take to solve $Lv = b$ in this case? Ignore the time it takes to build the matrix L. Print out the Matlab code.

c) (5 pts.) Write the same code as above, this time forming L without using the *sparse* command. How long does it take to solve $Lv = b$ in this case? Ignore the time it takes to build the matrix L. Compare now the solving times, for the code you just wrote and the code in b), if you also take into account the time it takes to build the matrix. Is there any difference? What is this phenomenon due to? (*Hint*: to solve a linear system in Matlab of the form $Ax = b$, given A and b, you can use the command line

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x= A\b
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provided that A is an invertible matrix, i.e. that the linear system has a unique solution!)