1. (9 points total)

Let

\[ f(x) = \begin{cases} 
\text{positive delta spike at even multiples of } \pi \\
\text{negative delta spike at odd multiples of } \pi 
\end{cases} \]  

(see figure below)

\[ \begin{array}{ccccccccccc}
\pi & -2\pi & 0 & -\pi & 2\pi & 3\pi & 5\pi & 6\pi & 7\pi & 8\pi \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
-7\pi & -6\pi & -5\pi & -3\pi & -\pi & \pi & 2\pi & 4\pi & 5\pi & 7\pi \\
\end{array} \]

a. (2 points) Is \( f(x) \) even or odd?

Even
b. (7 points) Compute a real Fourier series for $f(x)$ in simplest form.

$$f(x) = \frac{2}{\pi} [\cos x + \cos 3x + \cos 5x + ...]$$
2. (14 points total) This problem concerns the symmetric, circulant $4 \times 4$ matrix

\[
C = \begin{pmatrix}
a & b & c & b \\
b & a & b & c \\
c & b & a & b \\
b & c & b & a \\
\end{pmatrix}
\]

a. (6 points) Write down four (maybe not distinct) real eigenvalues of $C$ in simplest form. Write three separate conditions on $a, b, c$ which will make the matrix $C$ singular.

Eigenvalues are $a + 2b + c, a - 2b + c, a - c$ and $a - c$ again. The conditions are $a + 2b + c = 0$, $a = c$, and $a + c = 2b$. 
b. (8 points) Use the language of convolution and discrete Fourier transforms to derive how many of the 216 possible rolls of three ordinary dice sum to an exact multiple of 4? Hint: Something about a 4x4 DFT, \((-1 \pm i)^3 = 2 \pm 2i\). and (1 2 2 1). (One die face is a multiple of four, two are 1 mod 4, two are 2 mod 4, and one is 3 mod 4).

\[
\text{ft}([1 2 2 1]) = [6 -1-i 0 -1+i]
\]
If we cube this, and ifft, we get 55, 55, 53, 53, respectively.
3 (10 points total) The semicircle function is

\[ f(x) = \begin{cases} \sqrt{1-x^2} & \text{on } [-1, 1] \\ 0 & \text{otherwise} \end{cases} \]

3a. (5 points) We want to compute the integral Fourier transform \( \hat{f}(k) \). Suppose you find in a table of integrals that

\[ \int_{-1}^{1} \frac{-x}{\sqrt{1-x^2}} e^{-ikx} = \pi i J_1(k), \]

where \( J_1(k) \) is a Bessel function, and you remember that the derivative of \( \sqrt{1-x^2} \) is \( \frac{-x}{\sqrt{1-x^2}} \). Can you compute \( \hat{f}(k) \)? (Hint: There is no need to understand anything about Bessel functions though \( J_1 \) may appear in the answer.)

\[ \hat{f}(k) = \pi J_1(k)/k. \]

3b. (5 points) Compute explicitly for the function above the value of \( E \) in

\[ E = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk. \]

\[ E = 8\pi/3 \]