Quiz 3

18.085 (Prof. Edelman)

May 12, 2010

PRINTED NAME:_____

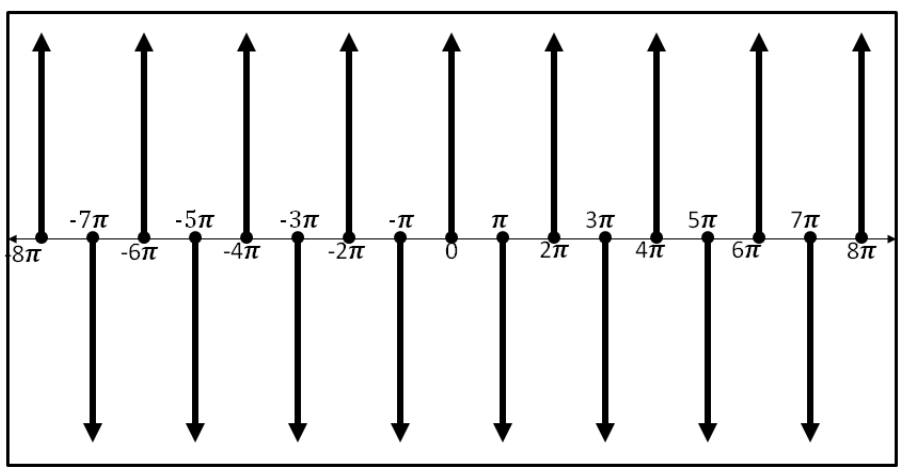
CLASS NUMBER (the number you use when submitting hw):_____

- Do all your work on these pages. No calculators or computers may be used. Notes and the text may be used. The point value (total is 33) of each subproblem is indicated.

1. (9 points total)

Let

$$f(x) = \begin{cases} \text{positive delta spike at even multiples of } \pi \\ \text{negative delta spike at odd multiples of } \pi \end{cases} \qquad \text{(see figure below)}$$



a. (2 points) Is $f(x)$ even or odd?

Even

b. (7 points) Compute a real Fourier series for $f(x)$ in simplest form.

$$f(x) = \frac{2}{\pi}\left[\cos x + \cos 3x + \cos 5x + ...\right]$$

2. (14 points total) This problems concerns the symmetric, circulant 4×4 matrix

$$C = \begin{pmatrix} a & b & c & b \\ b & a & b & c \\ c & b & a & b \\ b & c & b & a \end{pmatrix}$$

a. (6 points) Write down four (maybe not distinct) real eigenvalues of C in simplest form. Write three separate conditions on a, b, c which will make the matrix C singular.

Eigenvalues are $a+2b+c, a-2b+c, a-c$ and $a-c$ again. The conditions are $a+2b+c = 0$, $a = c$, and $a + c = 2b$.

b. (8 points) Use the language of convolution and discrete Fourier transforms to derive how many of the 216 possible rolls of three ordinary dice sum to an exact multiple of 4? Hint: Something about a 4x4 DFT, $(-1 \pm i)^3 = 2 \pm 2i$. and (1 2 2 1). (One die face is a multiple of four, two are 1 mod 4, two are 2 mod 4, and one is 3 mod 4).

fft([1 2 2 1])=[6 -1-i 0 -1+i]
If we cube this, and ifft, we get 55,55,53,53, respectively.

3 (10 points total) The semicircle function is

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{on } [-1,1] \\ 0 & \text{otherwise} \end{cases}.$$

3a. (5 points) We want to compute the integral Fourier transform $\hat{f}(k)$. Suppose you find in a table of integrals that

$$\int_{-1}^{1} \frac{-x}{\sqrt{1-x^2}} e^{-ikx} = \pi i J_1(k),$$

where $J_1(k)$ is a Bessel function, and you remember that the derivative of $\sqrt{1-x^2}$ is $\frac{-x}{\sqrt{1-x^2}}$. Can you compute $\hat{f}(k)$? (Hint: There is no need to understand anything about Bessel functions though J_1 may appear in the answer.)

$$\hat{f}(k) = \pi J_1(k)/k.$$

3b. (5 points) Compute explicitly for the function above the value of E in

$$E = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk.$$

$$E = 8\pi/3$$