Quiz 2

18.085 (Prof. Edelman)

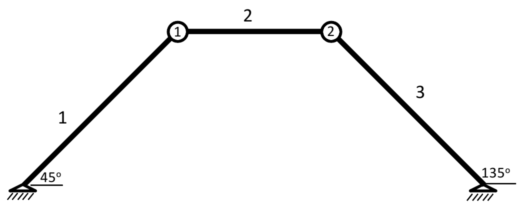
April 9, 2010

PRINTED NAME:____

CLASS NUMBER (the number you use when submitting hw):____________________

- Do all your work on these pages. No calculators or computers may be used. Notes and the text may be used. The point value (total is 33) of each subproblem is indicated.

1. (14 points total)



This problem concerns the truss above with bars labeled 1,2, and 3. Nodes 1 and 2 in the circles have displacement vectors (u_1^H, u_1^V) and (u_2^H, u_2^V), respectively. The other two nodes are fixed.

a. (5 points) Find the elongation matrix A so that

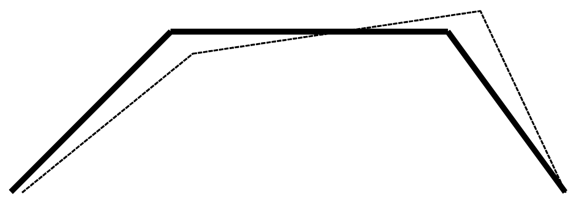
$$e = A \begin{pmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \end{pmatrix},$$

describes the stretch in bars 1, 2, and 3.

$$A - \begin{pmatrix} \sqrt{1/2} & \sqrt{1/2} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{1/2} & \sqrt{1/2} \end{pmatrix}$$

b. (5 points) What are the nullspace solutions to Au=0? Sketch the motion(s) corresponding to these solutions.

$$c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$



c. (1 points) How many bars do you need to add to eliminate these motions?

1 bar

d. (3 points) Where would you add any new bars, and what is the new matrix A?

Two good choices: 1) drop a bar vertically creating a new fixed node at the "floor" or 2) add a bar from say the fixed node on the left to node 2. These give

$$A = \begin{pmatrix} \sqrt{1/2} & \sqrt{1/2} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{1/2} & \sqrt{1/2} \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ or } A = \begin{pmatrix} \sqrt{1/2} & \sqrt{1/2} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{1/2} & \sqrt{1/2} \\ 0 & 0 & \cos\theta & \sin\theta \end{pmatrix}.$$

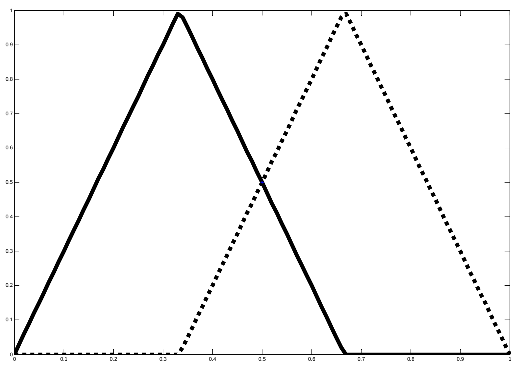
This could be with minus signs. The θ is an unknown angle, which is most certainly less than 45^o.

2. (10 points total) This problems concerns the Boundary Value Problem

$$\begin{cases} -u''(x) = \delta(x - 1/2), \\ u(0) = u(1) = 0. \end{cases}$$

a. (5 points) If we let $h = 1/3$, and use the piecewise linear hat functions ϕ_1 and ϕ_2 as in your text, we can set up a finite element equation $KU = F$. Calculate the vector F. (Note as usual the hat functions are used both for U and V.)

The hat functions are on $[1/3, 2/3]$ and $[2/3, 1]$ respectively. A quick sketch shows these hat functions each have the value $1/2$, at $x = 1/2$ so the integral against the



the delta functions are both $1/2$. Thus

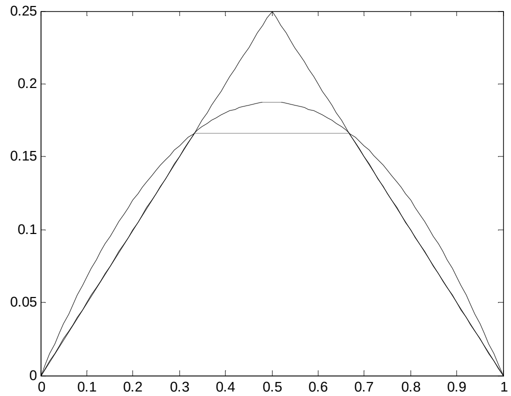
$$F = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

b. (5 points) The finite element solution to the above problem gives the same answer as a finite element solution to a Boundary Value Problem that does not involve any delta functions. Write down that Boundary Value Problem fully. Explain briefly but convincingly why the solution is the same.

The triangles in the figure above have area 1/3 each. Thus we need to set 3 times 1/2 or 3/2 as a constant right hand side, to get the same F. The problem is

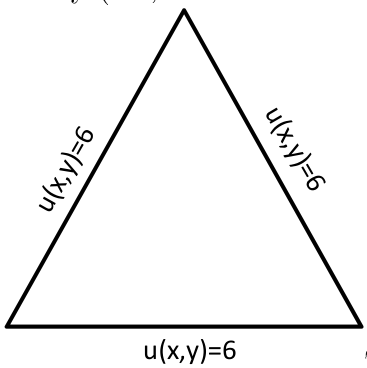
$$\begin{cases} -u''(x) = 3/2, \\ u(0) = u(1) = 0. \end{cases}$$

The solution to the original problem is a triangle, this new problem is a parabola, and the finite element solution is a trapezoid, and they all go through (1/3,1/6) and (2/3,1/6). At the points x=1/3 and x=2/3, the problems all look the same.


0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

3. (9 points total) These are three separate problems related to solutions to Laplace's equation.

a. (2 points) Find a solution u to Laplace's equation inside an equilateral triangle whose boundary (i.e., the three sides) has the value $u(x,y) = 6$.

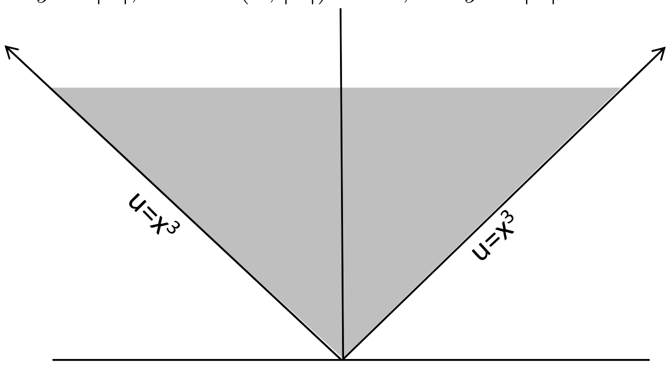


The function u(x,y)=6 in the interior is an obvious solution.

b. (3 points) Find a solution u, to Laplace's equation inside the unit circle whose boundary has the value $u_0(\theta) = \sin(2\theta + \pi/3)$.

This is just a rotation of the usual $\sin(2\theta)$ solution. The answer is $r^2 \sin(2\theta + \pi/3)$.

c. (4 points) Find a solution u, to Laplace's equation on the infinite "wedge" above the graph $y = |x|$, with $u(x, |x|) = x^3$, on $y = |x|$.



The cubic makes us think of the solution $u = x^3 - 3xy^2$. If $y = |x|$, then $u = -2x^3$ on the boundary. Therefore the correct solution is

$$u(x, y) = (-1/2)(x^3 - 3xy^2).$$