Quiz 1

18.085 (Prof. Edelman)

March 3, 2010

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- Do all your work on these pages. No calculators or computers may be used. Notes and the text may be used. The point value (total is 33) of each subproblem is indicated.

1. (17 points total)

a. (5 points) Write down a second order differential equation on $-1 \le x \le 1$, whose solution is $u(x) = |x|$. Be sure to include the boundary conditions as well as the differential equation. Be extra sure to get the coefficients exactly correct as there may not be partial credit.

The function $u(x) = |x|$ is piecewise linear with a jump in slope equal to $2 = 1 - (-1)$. Therefore the equation is

$$\frac{d^2u}{dx^2} = 2\delta(x),$$

with boundary conditions $u(-1) = u(1) = 1$.

b. (8 points) What is $u_2, u_3, u_4, \ldots, u_8$ exactly in the equation

$$K_9 \begin{pmatrix} 1/2 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ?$$

where K_9 is our special K, which we sometimes write as toeplitz([2 -1 0 0 0 0 0 0 0]).

The solution is piecewise linear with incremental steps $1/2$ on the left and $-1/2$ on the right. because $u_1 - u_0 = 1/2$, and $u_{10} - u_9 = -1/2$. From linearity, then the solution is

$$u = 1/2(1, 2, 3, 4, 5, 4, 3, 2, 1)^T.$$

c. (4 points) Give a mass-spring interpretation of the solution in b.

Imagine we have nine masses on ten springs with a fixed-fixed setup vertically under gravity. A load in the middle displaces all the msses downward with the maximum displacement at the middle, and a linearly decreasing displacement as we move towards the top and bottom. Notice that the springs above the middle are expanding, while the ones below the middle are compressing.

2. (16 points total) Consider fitting the data (-1, f_1),(0,f_2), $(1, f_3)$ to the curve $y = C + Dt^2$.

a. (4 points) What matrix equation $Au = f$ would we like to solve (but probably can not) for the best choice of C and D?

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

b. (3 points) Is $A^T A$ positive definite or only positive semi-definite?
\nicefrac{1}{2}
The columns of A are independent so $A^T A$ is positive definite.

c. (5 points) By hook or by crook, (but not with calculators or computers) decide whether $(A^TA)^{100}$ is very large, nearly 0, or something else? (Explain)

The diagonal elements of A^TA are 3 and 2 (sums of squares of columns of A). The trace is 5 so at least one eigenvalue is bigger than $5/2$. Therefore the 100th power is very large.

d. (4 points) Now consider fitting the same three points with the curve $y = C + Dt^2 + E(e^t + e^{-t} - 2)$. What is the new A?

Are the columns of this new matrix independent? (Explain your answer).

$$\begin{pmatrix} 1 & 1 & e + e^{-1} - 2 \\ 1 & 0 & 0 \\ 1 & 1 & e + e^{-1} - 2 \end{pmatrix}.$$

The third column is a scalar multiple of the second column so the columns are not independent. (When only sampling at -1,0,1, least squares can not distinguish between t^2 and $\cosh(t)$ in any meaningful way.)