1) (36 pts.) (a) Suppose \( u(x) \) is linear on each side of \( x = 0 \), with slopes \( u'(x) = A \) on the left and \( u'(x) = B \) on the right:

\[
  u(x) = \begin{cases} 
    Ax & \text{for } x \leq 0 \\
    Bx & \text{for } x \geq 0 
  \end{cases}
\]

What is the second derivative \( u''(x) \)? Give the answer at every \( x \).

(b) Take discrete values \( U_n \) at all the whole numbers \( x = n \):

\[
  U_n = \begin{cases} 
    An & \text{for } n \leq 0 \\
    Bn & \text{for } n \geq 0 
  \end{cases}
\]

For each \( n \), what is the second difference \( \Delta^2 U_n \)? Using coefficients \( 1, -2, 1 \) (notice signs!) give the answer \( \Delta^2 U_n \) at every \( n \).

(c) Solve the differential equation \(-u''(x) = \delta(x)\) from \( x = -2 \) to \( x = 3 \) with boundary values \( u(-2) = 0 \) and \( u(3) = 0 \).

(d) Approximate problem (c) by a difference equation with \( h = \Delta x = 1 \). What is the matrix in the equation \( KU = F \)? What is the solution \( U \)?
Solutions.

(a) $u''(x) = (B - A) \delta(x)$. This is not $B - A$ at $x = 0$.

(b) $\Delta^2 u_n = \begin{cases} B - A & n = 0 \\ 0 & n \neq 0 \end{cases}$

(c) $u(x) = \begin{cases} 3/5(x + 2) & -2 \leq x \leq 0 \\ 2/5(3 - x) & 0 \leq x \leq 3 \end{cases}$

(d) $K_{+++} = \begin{bmatrix} 2 & -1 & \cdots \\ -1 & 2 & -1 \\ \vdots & \vdots & \vdots \end{bmatrix}$

$U = \begin{bmatrix} 3/5 \\ 6/5 \\ 4/5 \\ 2/5 \end{bmatrix}$

$U$ lies right on the graph of $u(x)$
2) (24 pts.) A symmetric matrix $K$ is “positive definite” if $u^T Ku > 0$ for every nonzero vector $u$.

(a) Suppose $K$ is positive definite and $u$ is a (nonzero) eigenvector, so $Ku = \lambda u$. From the definition above show that $\lambda > 0$. What solution $u(t)$ to $\frac{du}{dt} = Ku$ comes from knowing this eigenvector and eigenvalue?

(b) Our second-difference matrix $K_4$ has the form $A^T A$:

$$K_4 = \begin{bmatrix}
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
\end{bmatrix}
= \begin{bmatrix}
2 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 \\
-1 & -1 & 2 & -1 \\
-1 & -1 & -1 & 2 \\
\end{bmatrix}
$$

Convince me how $K_4 = A^T A$ proves that $u^T K_4 u = u^T A^T A u > 0$ for every nonzero vector $u$. (Show me why $u^T A^T A u \geq 0$ and why $> 0$.)

(c) This matrix is positive definite for which $b$? Semidefinite for which $b$? What are its pivots??

$$S = \begin{bmatrix}
2 & b \\
b & 4 \\
\end{bmatrix}$$
Solutions. (8 pts each)

(a) “...show that $\lambda > 0$”

\[ Ku = \lambda u \]
\[ u^T Ku = \lambda u^T u \]
so
\[ \lambda = \frac{u^T Ku}{u^T u} > 0 \]

“What solution $u(t) \ldots$”

\[ u(t) = e^{\lambda t} u \]

(b) $u^T A^T Au = (Au)^T (Au) \geq 0$

The particular matrix $A$ in this problem has independent columns. The only solution to $Au = 0$ is $u = 0$. So $u^T A^T Au > 0$ except when $u = 0$. Other proofs are possible.

(c) Positive definite if $b^2 < 8$

Semidefinite if $b^2 = 8$

Pivots 2 and $4 - \frac{b^2}{7}$
3) (40 pts.) (a) Suppose I measure (with possible error) \( u_1 = b_1 \) and \( u_2 - u_1 = b_2 \) and \( u_3 - u_2 = b_3 \) and finally \( u_3 = b_4 \). What matrix equation would I solve to find the best least squares estimate \( \hat{u}_1, \hat{u}_2, \hat{u}_3 \)? Tell me the matrix and the right side in \( K \hat{u} = f \).

(b) What 3 by 2 matrix \( A \) gives the spring stretching \( e = Au \) from the displacements \( u_1, u_2 \) of the masses?

(c) Find the stiffness matrix \( K = A^T CA \). Assuming positive \( c_1, c_2, c_3 \) show that \( K \) is invertible and positive definite.
Solutions.

(a) (12 points)

\[ Au = b \]

\[ \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \]

\[ A^T \hat{u} = A^T b \]

\[ \begin{bmatrix} 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = \begin{bmatrix} b_2 - b_1 \\ b_3 - b_2 \\ b_4 + b_3 \end{bmatrix} \]

K 3pts 3pts

(b) (10 points) \( A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \)

(c) (16 points) \( A^T C A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 & -c_3 \\ -c_3 & c_3 \end{bmatrix} \)

Any of these proofs is OK:

1. \( \det = c_1 c_3 + c_2 c_3 > 0 \)
2. \( A^T C A \) always positive definite with independent columns in \( A \)
3. other ideas ...