1) (36 pts.)

(a) Suppose \( u(x) \) is linear on each side of \( x = 0 \), with slopes \( u'(x) = A \) on the left and \( u'(x) = B \) on the right:

\[
u(x) = \begin{cases} 
Ax & \text{for } x \leq 0 \\
Bx & \text{for } x \geq 0
\end{cases}
\]

What is the second derivative \( u''(x) \)? Give the answer at every \( x \).

(b) Take discrete values \( U_n \) at all the whole numbers \( x = n \):

\[
U_n = \begin{cases} 
An & \text{for } n \leq 0 \\
Bn & \text{for } n \geq 0
\end{cases}
\]

For each \( n \), what is the second difference \( \Delta^2 U_n \)? Using coefficients 1, \(-2\), 1 (notice signs!) give the answer \( \Delta^2 U_n \) at every \( n \).

(c) Solve the differential equation \(-u''(x) = \delta(x)\) from \( x = -2 \) to \( x = 3 \) with boundary values \( u(-2) = 0 \) and \( u(3) = 0 \).

(d) Approximate problem (c) by a difference equation with \( h = \Delta x = 1 \).

What is the matrix in the equation \( KU = F \)? What is the solution \( U \)?
2) (24 pts.) A symmetric matrix $K$ is “positive definite” if $u^T Ku > 0$ for every nonzero vector $u$.

(a) Suppose $K$ is positive definite and $u$ is a (nonzero) eigenvector, so $Ku = \lambda u$. From the definition above show that $\lambda > 0$. What solution $u(t)$ to $\frac{du}{dt} = Ku$ comes from knowing this eigenvector and eigenvalue?

(b) Our second-difference matrix $K_4$ has the form $A^T A$:

$$K_4 = \begin{bmatrix}
1 & -1 \\
-1 & 1 \\
1 & -1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & -1 \\
1 & -1
\end{bmatrix} = \begin{bmatrix}
2 & -1 \\
-1 & 2 \\
-1 & 2 \\
-1 & 2
\end{bmatrix}$$

Convince me how $K_4 = A^T A$ proves that $u^T K_4 u = u^T A^T A u > 0$ for every nonzero vector $u$. (Show me why $u^T A^T A u \geq 0$ and why $> 0$.)

(c) This matrix is positive definite for which $b$? Semidefinite for which $b$? What are its pivots??

$$S = \begin{bmatrix}
2 & b \\
b & 4
\end{bmatrix}$$
3) (40 pts.) (a) Suppose I measure (with possible error) $u_1 = b_1$ and $u_2 - u_1 = b_2$ and $u_3 - u_2 = b_3$ and finally $u_3 = b_4$. What matrix equation would I solve to find the best least squares estimate $\hat{u}_1, \hat{u}_2, \hat{u}_3$? Tell me the matrix and the right side in $K\hat{u} = f$.

(b) What 3 by 2 matrix $A$ gives the spring stretching $e = A u$ from the displacements $u_1, u_2$ of the masses?

(c) Find the stiffness matrix $K = A^TCA$. Assuming positive $c_1, c_2, c_3$ show that $K$ is invertible and positive definite.