18.085 Quiz 1 October 5, 2007 Professor Strang

Your PRINTED name is: SOLUTIONS **Grading** **1**
2
3

1) **(39 pts.)** With $h = \frac{1}{3}$ there are 4 meshpoints $0, \frac{1}{3}, \frac{2}{3}, 1$ and displacements u_0, u_1, u_2, u_3.

a) Write down the matrices A_0, A_1, A_2 with three rows that produce the first differences $u_i - u_{i-1}$:

A_0 has 0 boundary conditions on u

A_1 has 1 boundary condition $u_0 = 0$ (left end fixed)

A_2 has 2 boundary conditions $u_0 = u_3 = 0$.

b) Write down all three matrices $A_0^{\mathrm{T}} A_0, A_1^{\mathrm{T}} A_1, A_2^{\mathrm{T}} A_2$.

CROSS OUT IF FALSE / GIVE REASON BASED ON COLUMNS OF A!

$K_0 = A_0^{\mathrm{T}} A_0$ is (singular) (invertible) (positive definite)

Reason:

$K_1 = A_1^{\mathrm{T}} A_1$ is (singular) (invertible) (positive definite)

Reason:

c) Find all solutions $w = (w_1, w_2, w_3)$ to each of these equations:

$$A_0^{\mathrm{T}} w = 0 \qquad A_1^{\mathrm{T}} w = 0 \qquad A_2^{\mathrm{T}} w = 0$$

Solution.

a)

$$A_0 = \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

b)

$$A_0^{\mathrm{T}} A_0 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix} \qquad A_1^{\mathrm{T}} A_1 = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix} \qquad A_2^{\mathrm{T}} A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\downarrow$ singular

$\downarrow$ (invertible) (positive definite)

c)

$$A_0^{\mathrm{T}} w = \begin{bmatrix} -1 & & \\ 1 & -1 & \\ & 1 & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \longrightarrow \quad w = 0$$

$$A_1^{\mathrm{T}} w = 0 \quad \longrightarrow \quad w = 0$$

$$A_2^{\mathrm{T}} w = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \longrightarrow \quad w = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2) **(33 pts.)** a) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and unit eigenvectors y_1, y_2, y_3 of B. Hint: one eigenvector is $(1, 0, -1)/\sqrt{2}$.

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

b) Factor B into $Q\Lambda Q^{\mathrm{T}}$ with $Q^{-1} = Q^{\mathrm{T}}$. Draw a graph of the energy function $f(u_1, u_2, u_3) = \frac{1}{2}u^{\mathrm{T}}Bu$. This is a surface in 4-dimensional u_1, u_2, u_3, f space so your graph may not be perfect—**OK to describe it in 1 sentence**.

c) What differential equation with what boundary conditions on $y(x)$ at $x = 0$ and 1 is the **continuous analog** of $By = \lambda y$? What are the eigenfunctions $y(x)$ and eigenvalues λ in this differential equation? At which x's would you sample the first three eigenfunctions to get the three eigenvectors in part (a)?

Solution.

a) $y_1 = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}}$ has $By_1 = 0$ so $\lambda_1 = 0$ (check: trace =4)

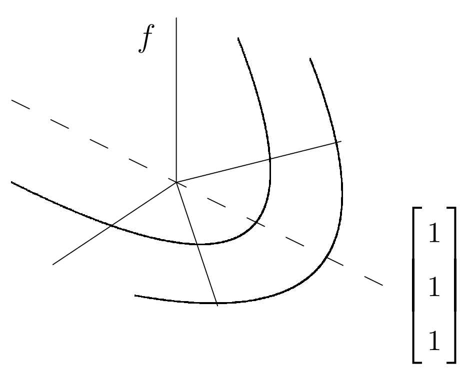
given vector $y_2 = \dfrac{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\sqrt{2}}$ has $By_2 = y_2$ so $\lambda_2 = 1$

y_3 is orthogonal to y_1, y_2 $\quad y_3 = \dfrac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}{\sqrt{6}}$ with $\lambda_3 = 3$

b) The orthonormal eigenvectors are the columns of Q (orthonormal gives $Q^{\mathrm{T}}Q = I$).

Then $B = Q\Lambda Q^{\mathrm{T}}$ with $Q = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{0}{\sqrt{2}} & \dfrac{-2}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 3 \end{bmatrix}$

The graph of $f = \frac{1}{2}u^{\mathrm{T}}Bu$ has a valley along the line of eigenvectors $u = (c, c, c)$. The surface goes up the orthogonal directions.

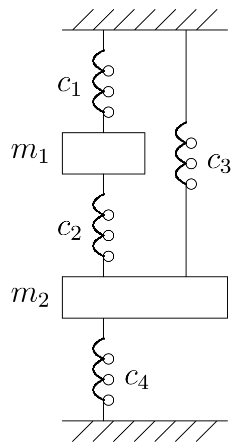


c) B is free-free so the equation is $-y'' = \lambda y$ with $y' = 0$ at $x = 0, 1$.

$$y_k = \cos k\pi x\,, \quad k = 0, 1, 2, \ldots$$

Sample at the points $x = \left(\frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ to get (a multiple of) the discrete eigenvectors y_1, y_2, y_3.

3) **(28 pts.)** The fixed-fixed figure shows $n = 2$ masses and $m = 4$ springs. Displacements u_1, u_2.



a) Write down the stretching-displacement matrix A in $e = Au$.

b) What is the stiffness matrix $K = A^{\mathrm{T}}CA$ for this system?

c) **Theory question about any $\boldsymbol{A^{\mathrm{T}}CA}$.** C is symmetric positive definite. What condition on A assures that $u^{\mathrm{T}}A^{\mathrm{T}}CAu > 0$ for every vector $u \neq 0$? Explain why this is greater than zero and where you use your condition on A.

Solution.

a)

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Au$$

b)

$$A^{\mathrm{T}}CA = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 + c_4 \end{bmatrix}$$

c) Write $u^{\mathrm{T}}A^{\mathrm{T}}CAu = (Au)^{\mathrm{T}}C(Au) = e^{\mathrm{T}}Ce$

Since C is positive definite, this is positive unless $e = 0$.

Condition on A: Independent columns.

Then $e = Au$ is zero only if $u = 0$.

So $A^{\mathrm{T}}CA$ is positive definite.