Homework Solutions PSet 1

1 Problem 1.1.2

$$U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

1)

$$T_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

2)

$$UU^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3)

$$\left(U^T U\right)^{-1} = U^{-1} \left(U^{-1}\right)^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2 Problem 1.2.5

```
K5 = 2*diag(ones(5,1)) - diag(ones(4,1),1) - diag(ones(4,1),1)'

K5 =

     2    -1     0     0     0
    -1     2    -1     0     0
     0    -1     2    -1     0
     0     0    -1     2    -1
     0     0     0    -1     2

inv(K5)
```

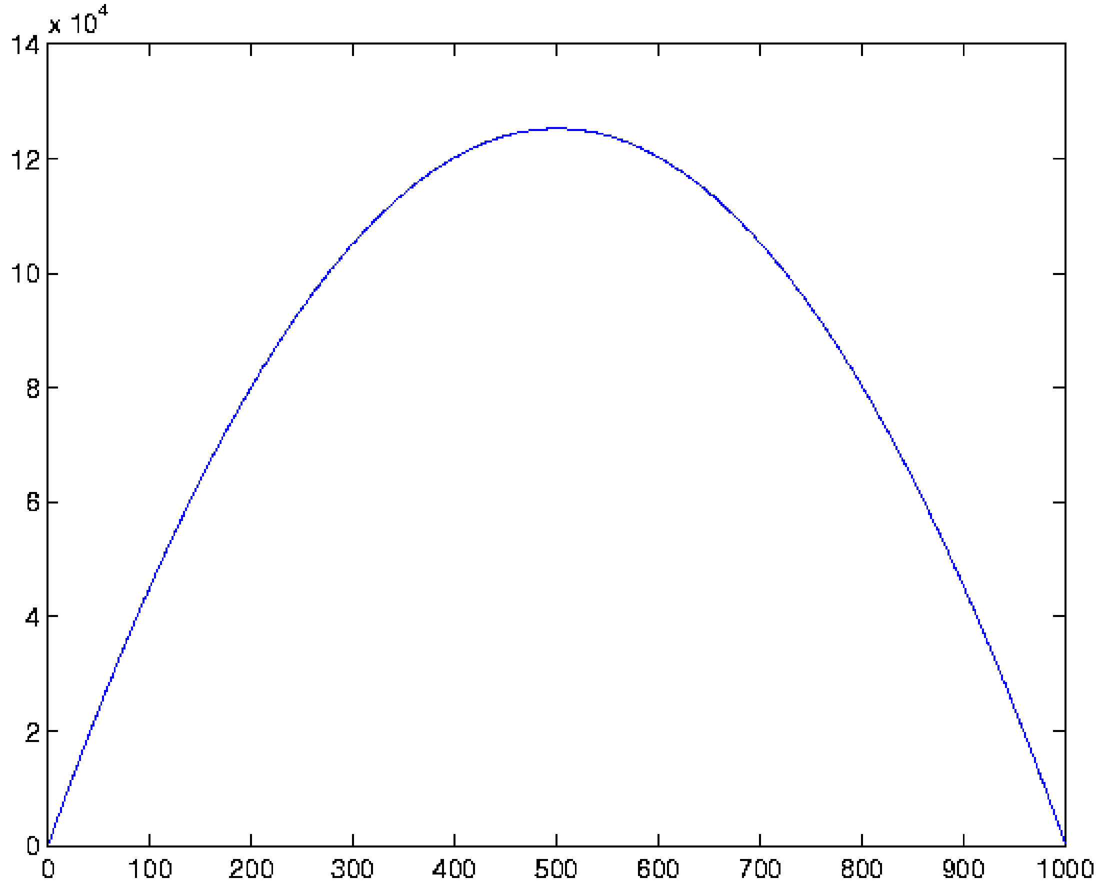
```
ans =

    0.8333    0.6667    0.5000    0.3333    0.1667
    0.6667    1.3333    1.0000    0.6667    0.3333
    0.5000    1.0000    1.5000    1.0000    0.5000
    0.3333    0.6667    1.0000    1.3333    0.6667
    0.1667    0.3333    0.5000    0.6667    0.8333

det(K5)

ans =

     6

det(K5)*inv(K5)

ans =

    5.0000    4.0000    3.0000    2.0000    1.0000
    4.0000    8.0000    6.0000    4.0000    2.0000
    3.0000    6.0000    9.0000    6.0000    3.0000
    2.0000    4.0000    6.0000    8.0000    4.0000
    1.0000    2.0000    3.0000    4.0000    5.0000
```

3 Problem 1.2.20

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 12 \\ 10 & 20 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 14 & 28 \end{bmatrix}$$

It will take n^3 multiplications to compute the product of two $n \times n$ matrices.

4 Problem 1.2.22



5 Problem 1.2.1

$$u''(x) = (B - A)\,\delta(x)$$

$$\left(\Delta^2 U\right)_n = u_{n+1} - 2u_n + u_{n-1}$$

$$
= \begin{bmatrix} \vdots \\ 0 \\ 0 \\ B-A \\ 0 \\ 0 \\ \vdots \end{bmatrix}
$$

6 Problem 1.2.2

$$
u(x) = \begin{cases} A(x+2) & x < 0 \\ B(x-3) & x \geq 0 \end{cases}
$$

We know the change in slope is -1 at $x = 0$ and that $A(x+2) = B(x-3)$ at $x = 0$, so $B = A - 1$ and $2A = -3B$, giving $B = \frac{-2}{5}$ and $A = \frac{3}{5}$.

$$
U = \begin{bmatrix} .6 \\ 1.2 \\ .8 \\ .4 \end{bmatrix}
$$

$$
KU = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} .6 \\ 1.2 \\ .8 \\ .4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

7 Problem 1.2.4

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & & \\ -1 & 1 & 0 & \\ & \ddots & \ddots & 0 \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \ddots & 1 & 1 & \\ 1 & \ddots & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} &= \begin{bmatrix} 1 & 0 & & \\ -1 & 1 & 0 & \\ & \ddots & \ddots & 0 \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_1 + u_2 \\ \vdots \\ \sum_{m=1}^{n} u_m \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ u_1 + u_2 - u_1 \\ \vdots \\ \sum_{m=1}^{n} u_m - \sum_{m=1}^{n-1} u_m \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}
\end{aligned}
$$

The 3×3 centered first difference matrix is

$$\Delta_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

The first and third columns are linearly dependent and we have

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$$

for any c. The rank is 2 so this describes the full nullspace. Similarly, for 5×5 the centered first difference matrix is

$$\Delta_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

It is also singular. It's rank is 4 and the nullspace is spanned by

$$\begin{bmatrix} c \\ 0 \\ c \\ 0 \\ c \end{bmatrix}$$

8 Problem 1.2.7

1) For $u = 1$, $u_1 = u_2 = u_{-1} = u_{-2} = 1$, so one the left side we have

$$\frac{-1 + 8 - 8 + 1}{12h} = 0$$

while $u'(x) = 0$, and all derivatives over the fifth are 0.

For $u = x^2$ we have

$$\begin{aligned} \frac{-\left(x+2h\right)^2 + 8\left(x+h\right)^2 - 8\left(x-h\right)^2 + \left(x-2h\right)^2}{12h} &= \\ \frac{\left(-1+8-8+1\right)x^2 + \left(-4+16+16-4\right)xh + \left(-4+8-8+4\right)}{12h} &= 2x \end{aligned}$$

and $u'\left(x\right) = 2x$, and all derivatives over the fifth are 0.

For $u = x^4$ we have

$$\frac{-\left(x+2h\right)^4 + 8\left(x+h\right)^4 - 8\left(x-h\right)^4 + \left(x-2h\right)^2}{12h}$$

and if one uses the binomial theorem to get

$$(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$$

and combines terms, we see that the left hand side equals $4x^3$, while $u'(x) = 4x^3$, and all derivatives over the fifth are 0. So in all three cases, the discrete difference actually equals the derivative of the function.

2) For the general case, we use

$$\begin{aligned} u(x+h) &= u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \ldots \\ &= \sum_{n=0}^{\infty} \frac{h^n}{n!} u^{(n)}(x) \end{aligned}$$

so that

$$\begin{aligned} u_2 &= \sum_{n=0}^{\infty} \frac{(2h)^n}{n!} u^{(n)}(x) \\ u_1 &= \sum_{n=0}^{\infty} \frac{h^n}{n!} u^{(n)}(x) \\ u_{-1} &= \sum_{n=0}^{\infty} \frac{(-h)^n}{n!} u^{(n)}(x) \\ u_{-2} &= \sum_{n=0}^{\infty} \frac{(-2h)^n}{n!} u^{(n)}(x) \end{aligned}$$

Then combining these in the difference equation we get

$$\frac{\sum_{n=0}^{\infty} \left[\frac{-(2h)^n + 8h^n - 8(-h)^n + (-2h)^n}{n!} u^{(n)}(x) \right]}{12h}$$

For any even n the numerator of the fraction inside the sum is 0, because in this case $(2h)^n = (-2h)^n$ and $(h)^n = (-h)^n$. When $n = 1$ the numerator is

$$-2h + 8h + 8h - 2h = 12h$$

When $n = 3$ it is

$$-8h^3 + 8h^3 + 8h^3 - 8h^3 = 0$$

and for $n = 5$ it is

$$-32h^5 + 8h^5 + 8h^5 - 32h^5 = -48h^5$$

thus the formula reduces to

$$u'(x) + \frac{-48}{5! * 12} h^4 u^{(5)}(x) + \ldots = u'(x) + \frac{-1}{30} h^4 u^{(5)}(x) + \ldots$$

as the problem states. $b = \frac{-1}{30}$.

9 Problem 1.2.10

We multiply

$$\begin{bmatrix} 1 & 0 & & \\ -1 & 1 & & \\ & \ddots & \ddots & 0 \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & & \\ 0 & -1 & 1 & \\ & \ddots & \ddots & -1 \\ & & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \\ & & 1 & -2 \end{bmatrix}$$

The bottom row is expressing the equality

$$u_{n-1} - 2u_n + u_{n+1} = u_{n-1} - 2u_n$$

which is equivalent to

$$u_{n+1} = 0$$

so this is a condition like $u = 0$. The top row is expressing

$$u_{-1} - 2u_0 + u_1 = -u_0 + u_1$$

which is equivalent to

$$u_{-1} - u_0 = 0$$

so this is a condition on a first difference, so it is an approximation of $u' = 0$

10 Problem 1.2.19

The centered finite second difference matrix with fixed-fixed boundary conditions should be a sum of the K matrix and the centered difference matrix Δ_0, so with $n = 4$ (and therefore $h = 1/5$) we have

$$K_0 = 25 \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & -1 & 0 & 1 \\ & & -1 & 0 \end{bmatrix} = \begin{bmatrix} 50 & -20 & 0 & 0 \\ -30 & 50 & 0 & 0 \\ 0 & -30 & 50 & -20 \\ 0 & 0 & -30 & 50 \end{bmatrix}$$

Likewise the forward finite second difference matrix should be $K + \Delta_+$ so

$$K_+ = \begin{bmatrix} 45 & -20 & 0 & 0 \\ -25 & 45 & -20 & 0 \\ 0 & -25 & 45 & -20 \\ 0 & 0 & -25 & 45 \end{bmatrix}$$

The true $u(x)$ is $x + Ae^x + B$ for some A and B. We know $u(0) = u(1) = 0$ so plugging these in we get

$$A + B = 0$$

$$1 + Ae + B = 0$$

Using these two equations we get

$$A = \frac{1}{1-e} \qquad B = \frac{-1}{1-e}$$

The following MATLAB output gives the rest of the solution:

```
K = 25*toeplitz([2 -1 0 0]);
D1 = 5*(diag(ones(3,1),1) - diag(ones(3,1),1)'); % Centered difference
D2 = 5*(diag(ones(3,1),1) - diag(ones(4,1))); % Forward difference
K1 = K + D1; % Centered second difference
K2 = K + D2; % Forward second difference
u = K1\ ones(4,1)
u =
    0.0621     0.1052     0.1199     0.0919
U = K2\ ones(4,1)
U =
    0.0782     0.1258     0.1355     0.0975
e=exp(1);
x=.2:.2:.8;
f = x + (1-e)^(-1)*exp(x) + (e-1)^(-1);
f'
ans =
    0.0711     0.1138     0.1215     0.0868
```

11 Problem 1.2.21

$u''(0) = f(0)$ and $u'(0) = 0$ so

$$\begin{aligned} u_0 - u_1 &= u(0) - u(h) \\ &= u(0) - \left(u(0) - \frac{1}{2} h^2 f(0) + O\left(h^3\right) \right) \\ &= \frac{1}{2} h^2 f(0) + O\left(h^3\right) \end{aligned}$$