18.085 Homework 8 Solutions

December 2, 2007

1 4.1.3

If you want, you can find the coefficients without doing any integrals by taking the real and imaginary parts of the series from problem 13.

$$\begin{aligned} a_{0,1,...} &= \{1/2, 2/\pi, 0, -2/3\pi, 0, 2/5\pi, ...\} \\ b_{1,2,...} &= \{0, 0, 0, ...\} \end{aligned}$$

2 4.1.10

You can find the Fourier series for $u_0(\theta)$ without doing any integrals by comparison with the Fourier series from section 4.1 equation (7). For the current problem,

$$u_0(\theta) = \frac{1}{2} + \frac{2}{\pi} sin\,\theta + \frac{2}{3\pi} sin\,3\theta + \frac{2}{5\pi} sin\,5\theta + ...$$

Equating this with section 3.4 equation (18) gives the a's and b's in the solution, section 3.4 equation (17).

$$u(r,\theta) = \frac{1}{2} + \frac{2}{\pi} r\,sin\,\theta + \frac{2}{3\pi} r^3\,sin\,3\theta + \frac{2}{5\pi} r^5\,sin\,5\theta + ...$$

$u(0,\theta) = \frac{1}{2}$.

3 4.1.13

From example 7,

$$c_{...,-3,-2,-1,0,1,2,3,4,5,...} = ..., -1/3\pi, 0, 1/\pi, 1/2, 1/\pi, 0, -1/3\pi, 0, 1/5\pi, ...$$

The right side of equation (31) adds up to $2\pi(\frac{1}{2}) = \pi$.

If h=2π instead, $F(x) = 1$ everywhere.

4 4.1.18

The general solution to the heat equation with the given boundary conditions and an even initial temperature function is

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} a_n \; cos\, nx \; e^{-n^2 t} \\ &= a_0 + a_1 \; cos\, x \; e^{-t} + a_2 \; cos\, 2x \; e^{-4t} + a_3 \; cos\, 3x \; e^{-9t} + ... \end{aligned}$$

Equating $u(x,0)$ with the Fourier series for the delta function given in equation (16) gives the a's.

$$u(x,t) = \frac{1}{2\pi} + \frac{1}{\pi} \; cos\, x \; e^{-t} + \frac{1}{\pi} \; cos\, 2x \; e^{-4t} + \frac{1}{\pi} \; cos\, 3x \; e^{-9t} + ...$$

5 4.2.1

For part a,

$$c_{mn} = \left\{ \begin{array}{cl} 1/4 & m = 0, n = 0 \\ -i/2\pi n & m = 0, n = 1, 3, 5, ... \\ -i/2\pi m & m = 1, 3, 5, ..., n = 0 \\ -1/\pi^2 mn & m = 1, 3, 5, ..., n = 1, 3, 5, ... \end{array} \right\}$$

For part b,

$$c_{mn} = \left\{ \begin{array}{cl} 1/2 & m = 0, n = 0 \\ -2/\pi^2 mn & m = 1, 3, 5, ..., n = 1, 3, 5, ... \end{array} \right\}$$

6 4.2.2

Functions that are 2π periodic and odd in both x and y.

$$\int_0^\pi \int_0^\pi (sin\, mx\ sin\, ny)\,(sin\, kx\ sin\, ly)\, dxdy = 0$$

unless $m = k$ and $n = l$.

7 4.2.20

$$\begin{array}{rcl} u_2[(pu_1')' + qu_1 + \lambda_1 w u_1] & = & 0 \\ -u_1[(pu_2')' + qu_2 + \lambda_2 w u_2] & = & 0 \\ \hline (pu_1')'u_2 - (pu_2')'u_1 & = & (\lambda_2 - \lambda_1) w u_1 u_2 \\ \int_a^b (pu_1')'u_2\, dx - \int_a^b (pu_2')'u_1\, dx & = & (\lambda_2 - \lambda_1) \int_a^b w u_1 u_2\, dx \\ pu_1'u_2|_a^b - \int_a^b pu_1'u_2'\, dx - pu_2'u_1|_a^b + \int_a^b pu_2'u_1'\, dx & = & (\lambda_2 - \lambda_1) \int_a^b w u_1 u_2\, dx \end{array}$$

The problem says the boundary terms ($pu_1'u_2|_a^b$ and $pu_2'u_1|_a^b$) are zero, and the integrals on the left side cancel with each other, leaving the right side equal to zero. If $\lambda_2 - \lambda_1 \neq 0$, this can only be true if $\int_a^b w u_1 u_2\, dx = 0$.

8 4.3.2

Each entry in row i of $\overline{F}$ is $\overline{w}^i$ times the previous entry. Each entry in row $N - i$ of F is $w^{N-i} = w^N w^{-i} = w^{-i} = \overline{w}^i$ times the previous entry also.

9 4.3.6

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & w & 0 \\ 0 & 0 & 1 & 0 & 0 & w^2 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -w & 0 \\ 0 & 0 & 1 & 0 & 0 & -w^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & w^2 & w^4 & 0 & 0 & 0 \\ 1 & w^4 & w^8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & w^2 & w^4 \\ 0 & 0 & 0 & 1 & w^4 & w^8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10 4.3.8

$$\begin{aligned} c &= (2,0,2,0) \\ c &= (2,0,-2,0) \end{aligned}$$

11 4.3.10

32nd; 128th.

12 4.3.15

$$\lambda_{1,2,3,4} = 1, i, -1, -i$$