18.085 Quiz 3 December 6, 2012 Professor Strang

Your PRINTED name is: SOLUTIONS

Your class number is 0

1.
2.
3.
4.

∞

(1) **(27 points)** This question is about the 2π–periodic one-sided pulse $f(x)$:

9 pts each part

$$f(x) = e^{-ax} \text{ for } 0 \le x \le \pi\,,\ f(x) = 0 \text{ for } -\pi < x < 0.$$

(a) Find the Fourier coefficients c_k in $f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$.

$$c_k = \frac{1}{2\pi}\int_0^{\pi} e^{-(a+ik)x}\,dx = \frac{-1}{2\pi}\,\frac{e^{-(a+ik)\pi} - 1}{a+ik}$$

(b) What is the decay rate of c_k as $k \to \infty$? What are all the singularities of $f(x)$ that produce this decay rate? You could graph $f(x)$.

The decay rate is $\frac{1}{k}$

$f(x)$ has TWO jumps (jump at $x=0$)

-3 for missing 2^{nd} jump (jump at $x = \pm\pi$)

(c) As $a \to 0$, what limit function $F(x)$ do you get? What are the Fourier coefficients of $F(x)$?

As $a \to 0$, $f(x)$ approaches $F(x) =$ square wave

$$= \begin{cases} 1 & 0 \le x \le \pi \\ 0 & -\pi < x < 0 \end{cases}$$

Fourier coefficients approach

$$c_k = \frac{-1}{2\pi}\,\frac{\left(e^{-ik\pi} - 1\right)}{ik} = \frac{-1}{2\pi}\,\frac{\left[(-1)^k - 1\right]}{ik}$$

$$= \frac{1}{\pi i k}\{0, 1, 0, 1, \ldots\}$$

(2) **(24 points)** Ordinary multiplication of the numbers 404 and 111 produces a convolution of the sequences 4, 0, 4 and 1, 1, 1.

6 points

(a) What sequence comes out of that convolution (not cyclic)?

Where does $1+1+1=3$ multiplied by $4+0+4=8$ appear in the answer to part (*a*)? Maybe this is a check that 3rd graders could use!

$4,0,4 * 1,1,1 = 4,4,8,4,4$

CAN DO THIS BY MULTIPLYING $(4+4=8)$ times $(1+1+1=3)$ is $(4+4+8+4+4) = 24$

$(4+4z^2)(1+z+z^2) = (4+4z+8z^2+4z^3+4z^4)$

2 points for this

(b) The solution to the differential equation $dy/dx = ay + s(x)$ with $y(0)=0$ is

8 points

$$y(x) = \int_0^x e^{a(x-t)} s(t)\,dt = e^{ax} * s(x)$$

not needed in my method

$\boxed{\text{Compute } dy/dx}$ to verify that this $y(x)$ solves the differential equation.

$y(x) = e^{ax}\left(\int_0^x e^{-at} s(t)\,dt\right)$ = product of two functions of x / use product rule

$$\frac{dy}{dx} = \underbrace{a e^{ax} \int_0^x e^{-at} s(t)\,dt}_{\text{THIS IS } ay} + \underbrace{e^{ax} \boxed{e^{-ax} s(x)}}_{\text{THIS IS } s(x)}$$

Fundamental Theorem of Calculus: Derivative of Integral gives original function

$\boxed{\text{Must use this somewhere since we don't know } s(x)}$

(c) If the source term is $s(x) = e^{bx}$, compute $y(x)$.

$$y(x) = \int_0^x e^{ax} e^{-at} e^{bt}\,dt$$

$$= e^{ax} \left. \frac{e^{(b-a)t}}{b-a} \right|_{t=0}^{x}$$

$$= \frac{e^{ax} e^{(b-a)x} - e^{ax}}{b-a} = \frac{e^{bx} - e^{ax}}{b-a}$$

8 pts

(3) (27 points)

9 pts

(a) What is the cyclic convolution $c \circledast d$ of those sequences $c = (4, 0, 4)$ and $d = (1, 1, 1)$ (with $N = 3$)?

$$\begin{array}{ccccc} & & 4 & 0 & 4 \\ & & 1 & 1 & 1 \\ \hline & & 4 & 0 & 4 \\ & 4 & 0 & 4 & \\ 4 & 0 & 4 & & \\ \hline 4 & 4 & 8 & 4 & 4 \end{array}$$

$(4,0,4) \circledast (1,1,1) = (8,8,8)$

You could transform the other way using F. Then $N=3$ appears

(b) That convolution transforms to a component by component multiplication $f .* g$. What are the vectors f and g, and what is $f .* g$ and where is the factor $N = 3$?

I like $f = Fc = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4+4w^2 \\ 4+4w^4 \end{bmatrix} = f$

$g = Fd = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = g$ Then $f .* g = \begin{bmatrix} 24 \\ 0 \\ 0 \end{bmatrix}$ ANSWER

INVERTING F gives $\frac{1}{3} \bar{F} \begin{bmatrix} 24 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$ ✓ DID NOT ASK FOR F^{-1} No ANSWER REQUIRED ABOUT $N=3$

N

(c) What 3 by 3 circulant matrix C gives cyclic convolution with $c = (4, 0, 4)$?

$$C \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = (4,\ 0,\ 4) \circledast (d_0,\ d_1,\ d_2).$$

$C = \begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 4 \\ 4 & 0 & 4 \end{bmatrix}$ note determinant $= 64 + 64 \neq 0$

c shift c cyclically

Show that the columns of the Fourier matrix F_3 are eigenvectors of C, and find the eigenvalues. Is C an invertible matrix?? YES ALL $\lambda \neq 0$

Eigenvectors/eigenvalues

$$\begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 4 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{bmatrix} = \begin{bmatrix} 8 & 4+4w & 4+4w^2 \\ 8 & 4w+4w^2 & 4w^2+4w^4 \\ 8 & 4+4w^2 & 4+4w^4 \end{bmatrix}$$

$\lambda_1 = 8$ $\quad \lambda_2 = 4+4w$ $\quad \lambda_3 = 4+4w^2$

Check trace/not asked

sum of eigenvalues $= 16 - 4 = \underline{12}$

sum of diagonal of $C = 4+4+4 = \underline{12}$

$\boxed{\text{Only need } w^3 = 1 \quad 1 + w + w^2 = 1 + w^2 + w^4 = 0}$

$-9+9+6$

(4) (22 points) Suppose I want to use the Fourier integral transform to solve this fourth-order differential equation:

$$\frac{d^4u}{dx^4} = \delta(x) - 4\delta(x-1) + 6\delta(x-2) - 4\delta(x-3) + \delta(x-4).$$

(a) Take the transform of all terms to find $\widehat{u}(k)$.

$(+ik)^4 \, \hat{u}(k) = 1 - 4e^{-ik} + 6e^{-2ik} - 4e^{-3ik} + e^{-4ik}$

$(i^4 = 1) \qquad = (1 - e^{-ik})^4$ ← 2 extra points for this

Divide by k^4 to get $\hat{u}(k)$

(b) The solution $u(x)$ will have jumps in which derivative (?) at which points x (?).

Jumps of $1, -4, 6, -4, 1$ in the third derivative

at $x = 0, 1, 2, 3, 4$

(c) What is $u(x)$ between $x = 0$ and $x = 1$? Do you recognize $u(x)$?

$u = 0$ for negative x.
At $x = 0$ the 3rd derivative jumps by 1.

Then $u = \dfrac{x^3}{6}$ from $x=0$ to $x=1$

it is a cubic B-spline (with u, u', u'' all continuous)

It is zero for $x \le 0$ and $x \ge 4$