18.085 Computational Science and Engineering Nov 3,2009

Solutions to Problem Set 7

Instructor: Gilbert Strang

Note: Solutions for 3.1-1 and 3.2-5 can be found in the Opencourseware. Besides, solutions for 3.2-17,3.2-19 (Matlab practice) will not be provided here.

[3.1-2]

$w(x) = (1-x)f$, and

$$u(x) = \int_0^x \frac{w(s)}{c(s)} ds = \int_0^x \frac{(1-s)f}{(1-s)} ds = \int_0^x f ds = fx.$$

[3.1-9]

Let $u(x) = d_1 + d_2 e^{x/c} + x$. Then $u(0) = d_1 + d_2 e^0 = 0$ implies $d_1 = -d_2$. By $u(1) = d_1 - d_1 e^{1/c} + 1$, we have $d_1 = \frac{1}{e^{1/c}-1}$ and $d_2 = \frac{1}{1-e^{1/c}}$.

When c goes to 0, we have both d_1 and d_2 go to 0. While u goes to x, we have u' goes to 1. Then the boundary condition at $x = 0$ is the same as before but at $x = 1$ has a boundary layer.

[3.1-11]

By $-u'' = x$, u has the form of

$$u = -1/6x^3 + C_1 x + C_2.$$

By the condition of $u(0)$ and $u(1)$, we have $C_1 = 1/6$ and $C_2 = 0$. So we have

$$u = -1/6x^3 + 1/6x.$$

Then we have

$$F_1 = \int_0^1 x\phi_1(x)dx = \frac{1}{9}$$

and

$$F_2 = \int_0^1 x\phi_2(x)dx = \frac{2}{9}.$$

Also we have

$$K_{11} = \int_0^1 \frac{d\phi_1}{dx}\frac{d\phi_1}{dx} = 6,$$

$$K_{12} = K_{21} = \int_0^1 \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} = -3$$

and

$$K_{22} = \int_0^1 \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} = 6.$$

So the equations $KU = F$ becomes

$$\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 2/9 \end{pmatrix},$$

which has the solution $u_1 = 0.049$ and $u_2 = 0.062$.

By Matlab, the largest difference between the exact and approximate solutions happens at $x = 0.84$ with an error 0.0116.

[3.1-14]

The area under ϕ_5 is

$$\int_h^{2h} \phi_5(x)dx = \frac{h}{6}\phi_5(h) + \frac{4h}{6}\phi_5(\frac{3h}{2}) + \frac{h}{6}\phi_5(2h).$$

By the condition that $\phi_5(h) = \phi_5(2h) = 0$ and $\phi_5(\frac{3h}{2}) = 1$, we have

$$\int_h^{2h} \phi_5(x)dx = \frac{4h}{6} = \frac{2}{9}.$$

[3.1-17]

Since $U(x)$ is a combination of continuous functions, it is also continuous. While for $U'(x)$, we can not directly get the conlusion that it is not continuous because it is a combination of discontinuous functions. Instead, let $U(x) = \sum_1^6 a_i(x)\phi_i$, and by computation we see that at $x = \frac{1}{3}$ and $x = \frac{2}{3}$, there are jumps for $U'(x)$, therefore $U'(x)$ is not continuous.

[3.2-5]

(a) Fixed-fixed: $u(0) = u(1) - 0$.

$\phi_0^d(x)$ and $\phi_3^d(x)$ can be droped.

(b) Fixed-free: $u(0) = 1$, $u'(1) = 0$.

$\phi_0^d(x)$ and $\phi_3^s(x)$ can be droped.

[3.2-7]

2 and 8.