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Solutions to Problem Set 5

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Note: Solutions for 2.4-8 and 2.4-17 can be found in the Opencourseware.

[2.4-7]

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$$

$$K = A^TCA = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c_1 + c_2 + c_3 \end{pmatrix}$$

Ground node four, we get

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$K = A^TCA = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$$

So we have $\det(K) = c_1c_2c_3$.

[2.4-9]

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, A^TA = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Ground node one, we get

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, K = A^TA = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Then we have

$$K^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

with eigenvalues $3.5321, 2.3473, 1.0000, 0.1206$ and $\det(K) = 1$.

[2.4-11]
Put $c_i = 1$ in (18), we have

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

If we put $b = (0, 12, 0, 0, 0, 0, -4, 4, 0)^T$, and solve linear equations $Sx = b$, we get $x = (-1, 7, -4, -2, -1, 3, -2, -1, 3)^T$. Therefore, $(u_1, u_2, u_3) = (-2, -1, 3)^T$.

If we ignore $-4, 4$ in the vector b and only concerned with $12V$ source, i.e.,put $b = (0, 12, 0, 0, 0, 0, 0, 0, 0)^T$, we will get $x = (-3, 6, -3, -3, 0, 3, -3, 0, 3)$. Therefore, $(u_1, u_2, u_3) = (-3, 0, 3)^T$.

[2.4-14]
Suppose there are n edges, labeled $1, \ldots, n$, then we have

$$(\text{column i of } A)^T(\text{column j of } A) = A_{1i}A_{1j} + \cdots + A_{ni}A_{nj}.$$

If nodes i and j are connected by the kth edge,

$$A_{ki}A_{kj} = -1 = -w_{ij},$$

and this the only nonzero term in the above sum. On the other hand, if nodes i and j are not connected, then the above sum is $0 = -w_{ij}$.

[2.4-15]
Note that $K = nI - E$, so we can compute eigenvalues of K by E. Since the eigenvalues of E are $n-1, 0, \ldots, 0$, we have eigenvalues of K are $1, n, \ldots, n$. Then the eigenvalues of K^{-1} are the inverses of the eigenvalues of K^{-1}: $1, \frac{1}{n}, \ldots, \frac{1}{n}$.

[2.4-18]
Use Matlab,

```
N=3;
B=toeplitz([2 -1 zeros(1,N-2)]);
B(1,1)=1;
B(N,N)=1;
L=kron(B,eye(N))+kron(eye(N),B);
K=L(1:8,1:8)
f=[1 0 0 0 0 0 0 0]
inv(K)*transpose(f)
```

we get $(1.5000, 1.0000, 0.7500, 1.0000, 0.7500, 0.5000, 0.7500, 0.5000)$. Therefore, $u(1,1) = 1.5$.