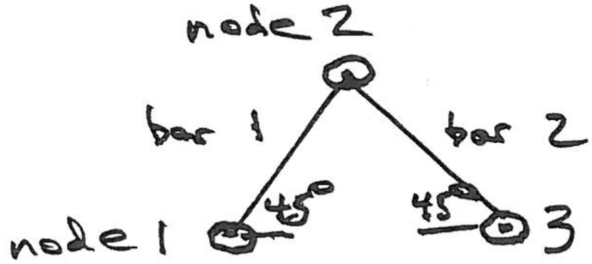
18.085 Quiz 2 November 5, 2012 Professor Strang

Your PRINTED name is: SOLUTIONS

Your class number is ____________

1. 20
2. 30
3. 20
4. 30



1. **(20 points)** This truss has only 2 bars and no supports. The bars are at 45° and −45° angles (so they meet at a 90° angle).

(a) Write down A, the "node displacement — bar stretch matrix" for this truss: to first order $e = Au$.

$$A = \begin{array}{cc} \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} & \begin{array}{l} \text{bar 1} \\ \text{bar 2} \end{array} \end{array}$$

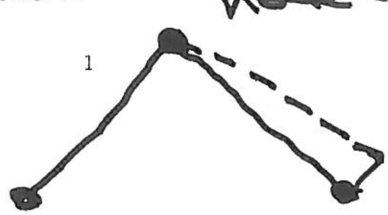
3 rigid motions and 1 mechanism

(b) Describe all solutions to $Au = 0$. How many independent solutions? 4 Draw any mechanisms (solutions u that are not rigid motions).

node 2 down $u_2^V = -1$
node 1 left $u_1^H = -1$
node 3 right $u_3^H = +1$
(or reverse all signs)

$$u = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Another mechanism could be rotation around node 2



$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

2. **(10+20 points)** Suppose $c(x) = 1$ for $0 \le x \le \frac{1}{3}$ and $c(x) = 2$ for $\frac{1}{3} < x \le 1$. (The material changed at $x = \frac{1}{3}$.) The fixed-fixed strong form is

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = 1 \text{ with } u(0) = 0 \text{ and } u(1) = 0.$$

(a) It looks as if dc/dx will produce a delta function on the right hand side (and not $f = 1$). Why does this delta function NOT happen?

The quantity $w = c(x)\frac{du}{dx}$ is continuous.

If $c(x)$ jumps, then $\frac{du}{dx}$ also jumps — to keep $c\frac{du}{dx}$ continuous.

(b) **Write down the weak form of this problem.** Use two linear functions ϕ_1, ϕ_2 (they are hat functions centered at $x = \frac{1}{3}$ and $\frac{2}{3}$) for an approximate solution $U(x) = U_1\,\phi_1(x) + U_2\,\phi_2(x)$. If the test functions are also $V_1 = \phi_1$ and $V_2 = \phi_2$, find the stiffness matrix K and load vector F in $KU = F$.

$$\int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^1 1 \cdot v(x)\,dx \quad \text{for all } v(x) \text{ with } \begin{pmatrix} u(0) = u(1) = 0 \\ v(0) = v(1) = 0 \end{pmatrix}$$

$$F_1 = \int_0^1 \phi_1\,dx = \text{area under hat} = \frac{1}{2} \times \frac{2}{3} \times 1 = \boxed{\frac{1}{3}} = F_2$$

$$K_{11} = \int c(x)\left(\frac{d\phi_1}{dx}\right)^2 dx = \frac{1}{h^2}\int_0^{2/3} c(x)\,dx = \frac{1}{h^2}\left(\frac{1}{3} + \frac{2}{3}\right)$$

$$K_{12} = K_{21} = \int c(x)\frac{d\phi_1}{dx}\frac{d\phi_2}{dx}\,dx = -\frac{1}{h^2}\int_{1/3}^{2/3} c(x)\,dx = -\frac{1}{h^2}\frac{2}{3}$$

$$K_{22} = \int c(x)\left(\frac{d\phi_2}{dx}\right)^2 dx = \frac{1}{h^2}\int_{1/3}^{1} c(x)\,dx = \frac{1}{h^2}\frac{4}{3}$$

Then $h = 1/3$: $K = 9\begin{bmatrix} 1 & -2/3 \\ -2/3 & 4/3 \end{bmatrix}$

3. **(20 points)**

(a) The vector field $\boldsymbol{v} = (e^{-x} \sin y, v_2)$ is the gradient of some function $u(x, y)$ under what conditions on $v_2(x, y)$? What is $u(x, y)$?

We need $\dfrac{\partial v_2}{\partial x} = \dfrac{\partial v_1}{\partial y} = e^{-x} \cos y$.

$u(x,y) = -e^{-x} \sin y$ (so that $v_1 = \dfrac{\partial u}{\partial x}$)

(b) The vector field $\boldsymbol{w} = (e^{-x} \sin y, w_2)$ has div $\boldsymbol{w} = 0$ for what functions $w_2(x, y)$? Then $\boldsymbol{w}$ comes from what stream function $s(x, y)$, with $\partial s/\partial x = w_2$ and $\partial s/\partial y = -w_1$.

Note Some function f has $f(x + iy) = u + i\,s$ but I'm not asking about f.

We need div w $= \dfrac{\partial}{\partial x}(e^{-x} \sin y) + \dfrac{\partial w_2}{\partial y} = 0$

$$-e^{-x} \sin y + \frac{\partial w_2}{\partial y} = 0$$

Then $w_2 = -e^{-x} \cos y$.

The stream function has

$\dfrac{\partial s}{\partial x} = w_2 = -e^{-x} \cos y$ so $s = e^{-x} \cos y$.

Check $\dfrac{\partial s}{\partial y} = -e^{-x} \sin y = -w_1$ ✓

$\boxed{\text{NOTE}}$ The book (which is always right) has the opposite sign for s: $w_1 = \dfrac{\partial s}{\partial y}$ $w_2 = -\dfrac{\partial s}{\partial x}$

BOTH ANSWERS ACCEPTED

$$\frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

4. (30 points)

(a) Write $\dfrac{1}{x+iy}$ as $u(x,y) + i\,s(x,y)$ so u and s are the real and imaginary parts.

Write $\dfrac{1}{re^{i\theta}}$ as $u(r,\theta) + i\,s(r,\theta)$ so u and s are the real and imaginary parts.

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\cos\theta - \frac{i}{r}\sin\theta$$

(b) Apply the Divergence Theorem $\iint (\operatorname{div} \boldsymbol{w})\,dx\,dy = \int (\boldsymbol{w}\cdot\boldsymbol{n})\,ds = \text{flux}$ to the vector field $\boldsymbol{w} = \operatorname{grad} u$ (same u) to show that the flux ~~into~~ through the circle $x^2+y^2=1$ equals the flux ~~out of~~ through the circle $x^2+y^2=2$. This depends on what fact about $\boldsymbol{w}$?

The divergence of w is ZERO between those circles / $\operatorname{div}(\operatorname{grad} u) = 0$ because u is coming from a function of $x+iy$

(c) At a point (x,y) on the unit circle, what is the direction of $\boldsymbol{w} = \operatorname{grad} u$? What is the value of $\boldsymbol{w}\cdot\boldsymbol{n}$? Integrate $\boldsymbol{w}\cdot\boldsymbol{n}$ (use x,y or r,θ) around the circle to find the flux coming out from the singularity at the center point $(0,0)$.

On the unit circle $x^2+y^2=1$ and $n = (x,y)$

$$\frac{du}{dx} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = -x^2+y^2 \quad [\text{IF } x^2+y^2=1] \qquad \frac{du}{dy} = \frac{-2xy}{(x^2+y^2)^2} = -2xy$$

$$\text{So } w\cdot n = (-x^2+y^2)x - 2xy(y) = -x^3 - xy^2 = -x(x^2+y^2) = -x$$

Integrate $-x$ around the circle to get $(r=1)$

$= -r\cos\theta$ $\qquad \int_0^{2\pi} \cos\theta\,d\theta = \boxed{0.}$

SEE PAGE 5

4(c) CONTINUED

An easier way is in polar coordinates:

$w \cdot n = (\text{grad } u) \cdot n = \dfrac{du}{dr}$ because n goes straight out

$u = \dfrac{\cos\theta}{r}$ has $\dfrac{du}{dr} = -\dfrac{\cos\theta}{r^2}$

Around the circle $r=1$, we integrate $-\cos\theta$ as before: $\boxed{0}$

* No flux out of the singularity at $r=0$
Interesting ...