18.085
Quiz SOLUTIONS *

Question 1

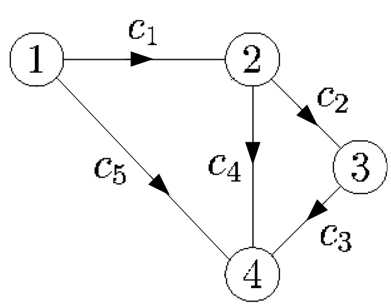
(**15 points**)

a Is $A^T A$ always positive definite for every matrix A?
NO!

b If there is a test, on A, what is it?
Test: A must have independent columns.

Question 2

(**5 points**)
What is the INCIDENCE matrix A for this graph?

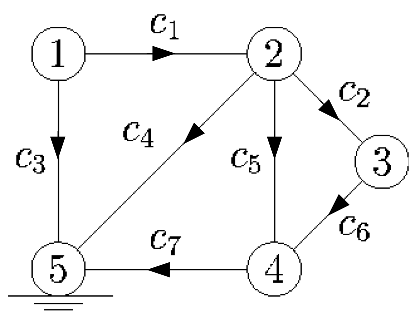


$$A = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \end{bmatrix}$$

*PRACTICE QUIZ. NOT for assessment.

Question 3

(**30 points**)
A network of nodes and edges is drawn. Edge number j has conductance $c_j > 0$. Node 5 is grounded (potential $u_5 = 0$). As usual, A is the incidence matrix and C is the diagonal matrix of conductances ($C_{j,j} = c_j$).



a List all positions (i, j) of the 4 by 4 matrix $K = A^T CA$ that have zero entries.
No edge between: nodes 1 and 3; nodes 1 and 4. So $K_{1,3} = K_{3,1} = 0 = K_{1,4} = K_{4,1}$.
Note: did not ask about $K_{unreduced}$ which would have $K_{3,5} = K_{5,3} = 0$

b What is row 1 of K?
Row 1 comes from edge 1:

$$[c_1 + c_3, -c_1, 0, 0]$$

c Find as many independent solutions as possible to Kirchhoff's Law $A^T y = 0$.

$$y_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

d What is 'the trick' that proves $u^T Ku \geq 0$ for every vector u?

$$u^T Ku = (u^T A^T)C(Au) = e^T Ce = c_1 e_1^2 + \cdots + c_m e_m^2.$$