Question 1 (matrix × vector approx. derivative)

a) Find $\frac{d^2}{dx^2} u(x)$ for:

i) $u(x) = 1$

ii) $u(x) = x$

iii) $u(x) = x^2$

iv) $u(x) = x^3$

b) $$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Find Ku for:

i) $u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ii) $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ iii) $u = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix}$ iv) $u = \begin{bmatrix} 1 \\ 8 \\ 27 \\ 64 \\ 125 \end{bmatrix}$

Question 2 (symmetric matrices have nice eigenvectors)

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Find:

i) the eigenvalues λ_1 & λ_2

ii) $\lambda_1 \lambda_2$

iii) $\lambda_1 + \lambda_2$

iv) trace(K)

v) det(K)

vi) the eigenvectors v_1 & v_2

vii) $v_1^T v_2$

viii) sketch v_1 & v_2 in the plane

Question 3 (rank and the "column x row" method)

$a_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

i) Find $a_1 a_1^T$.

(hint: $a_1^T = [1 \ \ -1]$)

ii) Find $a_2 a_2^T$.

iii) What is the <u>rank</u> of $a_1 a_1^T$?

iv) $A = [a_1 \ \ a_2]$.

Find AA^T.

(hint: try the "column x row" method.)

Question 4

(General solⁿ = homogeneous + particular matrix case, effect of changing boundary)

Let

$$C=\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad K=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad b=\begin{bmatrix} 10 \\ 10 \\ -10 \\ -10 \end{bmatrix}$$

i) Is C invertible?

ii) Find all solutions to $Cx = b$.

What is the homogeneous solution?

iii) Is K invertible?

iv) Find all solutions to $Kx = b$.

Question 5

(General solⁿ = homogeneous + particular)
effect of changing boundary

a) Find all solutions $u(x)$ to the O.D.E.

$$u'' + k^2 u = f$$

for

i) $f(x) = 0$

ii) $f(x) = x$

iii) $f(x) = \sin(kx)$

b) What is the homogeneous solution?

c) With $k = 2\pi$ and $f(x) = x$, find all solutions that satisfy the boundary conditions

$$0 = u(0) = u'(1) .$$

Question 6 (integration by parts ; transpose)

Let

$$A = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & \\ & & -1 & 1 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ h^2 \\ (2h)^2 \\ \vdots \\ ((N-1)h)^2 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ \cos(h\,\pi/2) \\ \cos(2h\,\pi/2) \\ \vdots \\ \cos((N-1)h\,\pi/2) \end{bmatrix}$$

- A is an $N \times N$ matrix with 1 on the diagonal and -1 below
- $h = 1/N$
- u is a vector with j^{th} component $(jh)^2$
- w is a vector with j^{th} component $\cos\left(jh\,\frac{\pi}{2}\right)$.

a) For $N = 10,\ 100,\ 1000$, find, to 2 decimals,

i) $(Au)^T w$

ii) $u^T (A^T w)$

b) Find $\int_0^1 2x \cos\left(\frac{\pi}{2} x\right) dx$

i) exactly with integration by parts, and

ii) to 3 decimals

Question 7 (simple Taylor series)

Let

$$f(x) = 1 - x + \frac{1}{2}x^2$$

$$E(x) = e^{-x} - f(x)$$

i) Find $E(x)$ to three significant figures for each x in the table:

x	1	0.1	0.01	0.001	0.0001
$E(x)$					

ii) Graph the points $(x, |E(x)|)$ on a loglog plot, and fit a straight line. Estimate the slope of the line to the nearest integer.

iii) Give a series expansion, in powers of x, for $E(x)$. What is the leading order?

Question 8 (Euler's formula $e^{i\theta}$)

$\lambda = -1 + 10i$. For $t > 0$, sketch

i) Real $(e^{\lambda t})$

ii) Imag $(e^{\lambda t})$

iii) $|e^{\lambda t}|$

Question 9

(Finite difference approx. to derivative and 2nd derivative ; 1st & 2nd order)

Let $f(x) = e^{\sin x}$

Then $f'(x) = \cos(x) f(x)$.

On a uniform grid of N points, on the interval $[0, 2\pi]$,

so $h = \dfrac{2\pi}{N}$

we represent the continuous functions f & f' by the discrete vectors

u_N with j^{th} component $f(hj)$, $j=1,\dots,N$.

V_N " " " $f'(hj)$, " .

For three matrices described next, and for $N = 10, 100, 1000, 10000$, compute:

$$e_N = A_N u_N - v_N$$

$$E_N = \| e_N \|_\infty$$

The three matrices A to try are the

- forward difference
- centered "
- backward "

approximations to the derivative:

$$\frac{1}{h}\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}, \quad \frac{1}{2h}\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ & & & & 1 \\ & & & -1 & 0 \end{bmatrix}, \quad \frac{1}{h}\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

These are <u>sparse</u> $N \times N$ matrices.

Notice how the magnitude of the first & last components of e compare to the magnitude of the rest of the components in the "middle" of e.

Throw away the first and last components by computing:

$$d_N = e_N(2:\text{end}-1)$$

$$D_N = \|d_N\|_\infty$$

(a) Graph (N, D_N) on a loglog plot.
Estimate the slope, to the nearest integer.
For each of the three matrices, hand in your graph and estimate of the slope.

(b) You see good convergence, but not always in the first component of e. For the centered difference matrix, give a new first row so that the first component is a good approx. to the derivative.
(Hint: $f(x)$ is periodic.)

(c) Repeat (a) and (b) for second derivatives.
For $N = 10, 100, 1000, 10000$, compute:

$$e_N = -\frac{1}{h^2} K_N u_N - w_N$$

$$d_N = e_N(2:\text{end}-1)$$

$$D_N = \| d_N \|_\infty$$

where w is a discretized second derivative,
and $K = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$ is our favourite matrix.

Graph (N, D_N) on a loglog plot.
Estimate the slope, to the nearest integer.
Hand in your graph and slope estimate.
Give a new first row of K so that the first component of e is a good approximation to the second derivative.