Problem Set 5

Due: November 19, 2015 (E18-366, 1:00pm)

Name (Print):

Instructions:

- Include a printed copy of this document with your solutions.
- Organize your work, in a neat and coherent way. Work without a clear ordering will receive very little credit.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive partial credit.
- Box your final answer to each problem.
- All the problems are worth the same amount (20 points/problem).

Problem 1 (T.F.):

(Suppose $\vec{F}$ is an irrotational force field, i.e. curl $\vec{F} = 0$, defined only on certain region of the plane, then in general we cannot conclude that $\vec{F}$ has a potential.

(1) Let

$$\vec{F}_1 = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right)$$

be a force field defined on the punctured plane (plane minus origin). Show that curl $\vec{F}_1 = 0$.

(2) Compute the work done by $\vec{F}_1$ for a particle looping around the unit circle counterclockwise.

(3) Is $\vec{F}_1$ a gradient vector field (conservative force field)? If yes, find a potential function.

(4) Do (1), (2) and (3) for

$$\vec{F}_2 = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right).$$

Problem 2 (T.F.):

(1) ([Section 3.3 Problem 18]) The curl operator is not invertible. Its nullspace contains all gradient fields $v = \text{grad } u$. Take the "determinant" of the 3 by 3 curl matrix in (29) to show that formally det(curl) = 0. Find two "vector potentials" S_1 and S_2 whose curl is $(2x, 3y, -5z)$.

(2) Recall that for a smooth function $f = f(x, y, z)$ vanishing at infinity, its Fourier transform $\widehat{f} = \widehat{f}(\xi_x, \xi_y, \xi_z)$ is a function defined by

$$\widehat{f}(\xi_x, \xi_y, \xi_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-i(x\xi_x + y\xi_y + z\xi_z)} \mathrm{d}x \mathrm{d}y \mathrm{d}z.$$

Similarly, for a vector field $g = (g_1, g_2, g_3)$, we define its Fourier transform (which is also a vector field) by taking Fourier transform componentwise

$$\widehat{g} = (\widehat{g_1}, \widehat{g_2}, \widehat{g_3}).$$

Assume we already know that (this can be easily proved by integration by part)

$$\widehat{f_x} = i\xi_x \cdot \widehat{f}, \quad \widehat{f_y} = i\xi_y \cdot \widehat{f}, \quad \widehat{f_z} = i\xi_z \cdot \widehat{f},$$

where f_x is the partial derivative of f with respect to x and $i = \sqrt{-1}$. Show that formally the Fourier transform of ∇ is $i\mathcal{E}$, where $\mathcal{E}$ is the so-called Euler vector field

$$\mathcal{E} = (\xi_x, \xi_y, \xi_z),$$

in the sense that

$$\widehat{\nabla f} = \widehat{\text{grad } f} = i\mathcal{E} \cdot \widehat{f},$$
$$\widehat{\nabla \times g} = \widehat{\text{curl } g} = i\mathcal{E} \times \widehat{g}.$$

(3) As we see in (2), the Fourier transform has the advantage that it transforms partial derivatives to algebraic operations (multiplication). Use Fourier transform to give an explanation for the fact that the nullspace of the curl operator are exactly those gradient fields.

Problem 3 (A.L.):
From the text, the Green's function of the Laplacian on the plane with source at $w = u + iv$ is

$$G(z, w) = -\frac{1}{2\pi} \log |z - w|, \tag{1}$$

where $z = x + iy$ and $|z|^2$ is defined as $x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}$ where $z = x - iy$. I.e.

$$\Delta G(z, w) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) G(z, w) = \delta(z - w). \tag{2}$$

From this formula compute the Green's function of the Laplacian on the upper half plane (for $y > 0$ when $z = x + iy$). Hint 1: remember you want a function that is zero on the boundary (the line where $y = 0$) but still satisfies $\Delta G(z, w) = \delta(z - w)$. Hint 2: compare the values of $G(z, w)$ and $G(z, \bar{w})$ when $z = x$, are they equal? Remember, geometrically $\bar{w}$ is at $u - iv$, that is, reflecting w across the real line gives $\bar{w}$. Can you combine $G(z, w)$ and $G(z, \bar{w})$ to get something that is zero on the line $z = x$?

Problem 4 (A.L.):
Check that for the polynomial $f(x, y) = x^2 - y^2$, we have the following property:

$$\frac{1}{2\pi r} \int_{C_{r,(u,v)}} f(\gamma) d\gamma = f(u, v), \tag{3}$$

where $C_{r,(u,v)}$ is the circle of radius r centered at the point (u, v). That is, show that the average of f on the circle $C_{r,(u,v)}$ is the value f at the center of the circle. This property is satisfied by every solution to Laplace's equation, it is called the mean value property. Hint: use polar coordinates and the fact that every point on $C_{r,w}$ is of the form $(u+r\cos\theta, v+r\sin\theta)$ where $0 \leq \theta \leq 2\pi$.

Problem 5 (S.C.):
(1) Recall that the Kronecker product $k(A, B)$ of two square matrices A and B of size n is the square matrix of size n^2 where a_{ij} is replaced by the block $a_{ij}B$. Using block multiplications, show that $k(A, B)k(C, D) = k(AC, BD)$. Deduce what is $k(A, B)^{-1}$ when both A and B are invertible ? What is $k(A, B)^T$?

(2) Let us consider the unit square in the plane and discretize it in a grid of increment $\frac{1}{n+1}$. Given boundary conditions on the nodes, we want to solve the Laplace equation $\Delta u = -4$. Recall how this problem translates into a linear equation $MU = F$ using finite differences. What are the nice properties satisfied by M ? In particular, give its eigenvectors and corresponding eigenvalues. Recall the 3 steps required to solve $MU = F$ using Fast Fourier Transform (FFT)

(3) We now use the following boundary conditions : $u(0, \frac{i}{n+1}) = u(\frac{i}{n+1}, 0) = 1 - (\frac{i}{n+1})^2$ and $u(1, \frac{i}{n+1}) = u(\frac{i}{n+1}, 1) = -(\frac{i}{n+1})^2$. Let $n = 5$. What is F_{ij} ? Using FFT with *MATLAB*, compute U. What is U_{33} ? If we take n to be $2k-1$ for some integer k what will U_{kk} tend to ?

Problem 6 (S.C.):
(1) Let D be a domain divided into triangles t. Let S be the set of all the vertices of these triangles and the midpoints of their edges and f a function from S to $\mathbb{R}$. For any triangle t in D, let ϕ_t be a quadratic polynomial in x and y which agrees with $f(s)$ for all $s \in S \cap t$. Why can we do it ? If t and t' share an edge e, do ϕ_t and $\phi_{t'}$ agree on e ?

(2) Using Persson's code (page 303 of the book or at http://math.mit.edu/ gs/cse/codes/femcode.m), solve the equation $\Delta u = -1$ with $u = 0$ on the standard unit square. Take $h = \frac{1}{4}$. Print the mesh information in the lists p and t and b (boundary node numbers). Display Kb and Fb and Ub.