

Problem Set 3

Due: October 15, 2015 (E18-366, 1:00pm)

Name (Print):

Instructions:

- Include a printed copy of this document with your solutions.
- Organize your work, in a neat and coherent way. Work without a clear ordering will receive very little credit.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive partial credit.
- Box your final answer to each problem.
- All the problems are worth the same amount (20 points/problem).

Problem 1 (A.L.):

Recall the formula in section 1.6 for Newton's method for minimization. Namely if $f(x_1, \dots, x_n)$ is a real-valued function on n -dimensional space then an iterative scheme to minimize f is as follows:

$$x^{i+1} = x^i - [H_f(x^i)]^{-1} \nabla f(x^i), \quad i \geq 0,$$

where $x^i = (x_1^i, \dots, x_n^i)$ is the i -th iteration of the algorithm, further ∇f is the gradient of f and H_f is the matrix of 2nd derivatives — called the Hessian of f .

(1) Let $g(x) = (x + 1)^2(x - 1)^2$, consider the function $f(x, y) = g(x)g(y)$. Compute ∇f and H_f .

(2) Run the Newton method in MATLAB for the following points (run them 10 steps each and print the output for each):

$$v_1 = (10, 0), \quad v_2 = (0, -10) \quad v_3 = (-3, -2) \quad v_4 = (2, 4). \quad (1)$$

Please print your output in a table with the values of the iteration for each v_1, \dots, v_4 on the row and the iteration on the column. Which points does each appear to converge to?

(3) Compute by hand the Newton method for the initial point equal to $(0, 0)$. What happens? Is the value at a local minimum here?

Problem 2 (T.F.):

Consider the fixed-free system in Section 2.1 consisting of 3 identical springs with $c = 1$ and 3 massive objects with $f = (1, 2, 3)$. You are allowed to permute the order of these objects in whatever way you like.

- (1) Use MATLAB to run over all the possible configurations (there are 6 of them) to find the configuration such that the total elongation is the smallest.
- (2) What is the pattern in (1) you find? Can you generalize it to the n -object case and give a proof?

Problem 3 (T.F.):

Consider the fixed-fixed system in Section 2.1 consisting of 4 identical springs with natural length l , stiffness c and 3 massive objects of mass m_1 , m_2 and m_3 respectively. Let $M = c \cdot l / g$ be a reference mass. Assume that the system will break if any of its springs has deformation strictly larger than $l/2$.

- (1) Find the threshold mass M' depending on M only, such that the following condition is satisfied. If $m_1 + m_2 + m_3 < M'$, no matter how m'_i 's are distributed, the system will not break. If $m_1 + m_2 + m_3 > M'$ instead, there is always a configuration of m_i 's that breaks the system.
- (2) Find the other threshold mass M'' depending on M only, such that the following condition is satisfied. If $m_1 + m_2 + m_3 > M''$, no matter how m'_i 's are distributed, the system breaks. If $m_1 + m_2 + m_3 < M''$ instead, there is always a configuration of m'_i 's such that the system will not break.

Problem 4 (S.C.):

- (1) Many physical phenomena are described by an equation of the form $x'' + wx = 0$. It is the case for the oscillation of a spring around its equilibrium position as discussed in G.Strang's book. Do you know any other field in physics where this equation appears?
- (2) From now on let us consider the equation

$$u'' + 4u = 0 \quad (*)$$

with initial condition $u(0) = 1$ and $u'(0) = 0$. What is the exact solution? What is the curve described by $t \rightarrow (u(t), u'(t))$ in the phase plane? Can you give its equation and its area? What is the period of revolution P around this curve? Plot the curve on *MATLAB* and hand it in

- (3) We can make (*) linear of first order by adding a variable v and considering the system

$$\begin{aligned} u' &= v \\ v' &= -4u \end{aligned}$$

What is the leapfrog discretization of this system? Write it in the form $\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = M(h) \begin{pmatrix} U_n \\ V_n \end{pmatrix}$ where $M(h)$ is a 2×2 matrix depending on the time increment h . After n steps, i.e. at time nh , the approximate solution is $M(h)^n \cdot X$ where $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If we choose $h = \frac{P}{N}$ for some N

the exact solution will come back to the vector X at time Nh . Using *MATLAB*, find an integer k such that $(X^T M(\frac{P}{N})^N X - 1)N^k$ has a nonzero limit (a real number, not infinity) when N grows to infinity.

(4) (Harder) Let us admit that the eigenvalues of $M(\frac{P}{N})$ are $1 + \frac{2iP}{N} + o(\frac{1}{N})$ and $1 - \frac{2iP}{N} + o(\frac{1}{N})$ with corresponding eigenvectors $v_N^+ = \begin{pmatrix} 1 \\ 2i \end{pmatrix} + o(1)$ and $v_N^- = \begin{pmatrix} 1 \\ -2i \end{pmatrix} + o(1)$ where $o(f)$ means negligible in comparison with f . Using the fact that $(1 + \frac{z}{N})^N \rightarrow e^z$ for any complex number z , conclude that $X^T M(\frac{P}{N})^N X \rightarrow_\infty 1$.

Problem 5 (S.C.):

Consider the equation $\frac{du}{dt} = Au(t)$ with

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

- (1) For which initial condition $u(0)$ will the solution be stationary ($u(t) = u(0)$ for all t) ?
- (2) Show that the norm is conserved: $\|u(t)\|^2 = \|u(0)\|^2$ for all t
- (3) The exponential of the matrix tA is defined as the series

$$Q = e^{tA} = I_3 + tA + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^n}{n!} + \cdots$$

Assuming that $e^A e^B = e^{A+B}$ if two matrices A and B commute check that Q is orthogonal, namely that $QQ^T = I_3$. Let X be a column vector. What is $\frac{d}{dt}(QX)$? Deduce the solution to the equation with initial condition $u(0) = X$.

Problem 6 (A.L.):

Problem 14 in Section 2.3. This problem projects $b = (b_1, \dots, b_m)$ onto the line through $a = (1, \dots, 1)$. We solve m equations $au = b$ in 1 unknown (by least squares).

- (1) Solve $a^T a \hat{u} = a^T b$ to show that \hat{u} is the average of the b_i .
- (2) Find $e = b - a\hat{u}$ and the variance $\|e\|^2$ and the standard deviation $\|e\|$.
- (3) The horizontal line $\hat{b} = 3$ is closest to $b = (1, 2, 6)$. Check that $p = (3, 3, 3)$ is perpendicular to e and find the projection matrix $P = A(A^T A)^{-1} A^T$.