# Problem Set 2

Due: October 1, 2015

- Include a printed copy of this document with your solutions.
- Organize your work, in a neat and coherent way. Work without a clear ordering will receive very little credit.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive partial credit.
- Box your final answer to each problem.

| Name (Print): |
|---|
| |

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| 6 | 20 | |
| **Total:** | 120 | |
| | **Total (%):** | |

**Problem 1 (S.C.):** Let $A$ be an invertible symmetric $n \times n$ matrix which admits a $LU$ factorization. Recall that $L$ is lower triangular, $U$ is upper triangular and that $A = LU$. In other words, one can find $n$ pivots for $A$ without exchanging rows.
(1) Prove that there exists a diagonal matrix $D$ such that $U = DL^T$
(2) Prove that $A$ is definite positive if and only if $X^TAX > 0$ for all column vectors $X$

**Problem 2 (S.C.):** Let

$$M = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix} \tag{1}$$

where $a$ is a real number. Find out for which values of $a$ $M$ is $LU$ factorizable, $PLU$ factorizable or non invertible. In the two first cases, give the $LU$ ( resp. $PLU$ ) factorization.

**Problem 3 (T.F.):** [Section 1.4 Problem 4]
(1) Solve the equation $-\mathrm{d}^2u/\mathrm{d}x^2 = \delta(x-a)$ analytically with **fixed-free** boundary conditions $u(0) = 0$ and $u'(1) = 0$.
(2) Draw the graphs of $u(x)$ and $u'(x)$.
(3) Solve the discrete problem with Matlab for $n = 2$, 3, 4, and 5 ($n$ is the number of interior points) and compare in one plot the results with the analytical solution when the load is located at $x = 1/3$. In the discrete problem, locate the load in the nearest node in each case.

**Problem 4 (T.F.):**
(1) Rigourously speaking, the delta function is not a function but a distribution (generalized

function): $\delta$ should be thought of as a linear functional on the space of smooth functions. That is, for any smooth function $f : \mathbb{R} \to \mathbb{R}$ vanishing at infinity, we assign the number $\delta(f) := f(0)$ to it. Formally, we can express this assignment by an integral

$$\delta(f) = f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x. \tag{2}$$

The delta function has a derivative $\delta'$ which is also a distribution. Find $\delta'(f)$ for any smooth function $f$ vanishing at infinity by a formal integration by part argument (cf. Section 1.4 Problem 15).

(2) We may also define $\delta'$ by the following procedure. Let

$$\rho_n(x) = \frac{n}{\sqrt{2\pi}} e^{-n^2x^2/2} \tag{3}$$

be the normal distribution $N(0, 1/n)$. Assuming that we can use $\rho_n$ to approximate the delta function:

$$\delta(x) = \lim_{n\to\infty} \rho_n(x) \tag{4}$$

in the sense that

$$\delta(f) = f(0) = \int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x = \lim_{n\to\infty} \int_{-\infty}^{\infty} \rho_n(x)f(x)\mathrm{d}x \tag{5}$$

for any smooth function $f$ vanishing at infinity. Then we can define $\delta'$ by setting

$$\delta'(f) = \int_{-\infty}^{\infty} f(x)\delta'(x)\mathrm{d}x := \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\rho_n'(x)\mathrm{d}x. \tag{6}$$

Show that this definition agrees with the one in (1).

(3) Use MATLAB to check

$$\delta(x) = \lim_{n\to\infty} \rho_n(x) \tag{7}$$

by verifying

$$f(0) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \rho_n(x)f(x)\mathrm{d}x \tag{8}$$

numerically, for $f(x) = \sin(\cos(x))$ and for $n = 1, 2, \ldots, 100$.

**Problem 5 (A.L.):** [Section 1.5 Problem 27]. Explain why $A$ and $A^T$ have the same eigenvalues. Show that $\lambda = 1$ is always an eigenvalue when $A$ is a Markov matrix, because each row of $A^T$ adds to 1 and the vector (?) is an eigenvector of $A^T$.

**Problem 6 (A.L.):** Let $Q$ be the following $n \times n$ matrix:

$$\frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix} = \frac{1}{3}\texttt{toeplitz}([1, 1, \texttt{zeros}(1, n-3), 1]). \tag{9}$$

This matrix is a Markov matrix (pardon, but here we use the convention that the row sum of $Q$ is one, rather than the row sum of $Q^T$ but in this case there is no difference) . If $\mu$ is a vector on $\mathbb{R}^n$ with positive components such that $\sum_{i=1}^n \mu_i = 1$, then $\mu$ represents a probability distribution on the set $\{1, \ldots, n\}$ ($\mu_i$ is the probability of picking $i$).

(1) Show that the vector $Q\mu$ is also a probability distribution in the same way that $\mu$ is (check each component $(Q\mu)_i$ is non-negative and the sum of all the components is 1). Remark: In the theory of Markov chains $Q$ is a **transition matrix** and $\mu$ is an **initial distribution** of the chain. $Q\mu$ represents the new probability distribution after one step of time.

(2) Show that $Q^t\mu \to \mathbf{1}/n$ as $t \to \infty$ where $\mathbf{1}$ is the vector of ones. Remark: this means that if you run the Markov chain for large time, $t$, the chain converges to the uniform distribution on $\{1, \ldots, n\}$ i.e. each $1, \ldots, n$ is equally likely no matter what your initial distribution $\mu$ is (!). Hint: recall that $Q$ symmetric implies $Q = \sum_{i=1}^N \lambda_i v_i v_i^T$, where $\lambda_i$ are the eigenvalues of $Q$ and $v_i$ are the orthonormal basis of eigenvectors $Qv_i = \lambda_i v_i$. Problem 5 and part (1) should be useful.

(3) Write some Matlab code to compute $\|Q^t\mu - \mathbf{1}/n\|$ for large $t$, where $\mu = e_1$ (i.e. the initial state of the Markov chain is deterministically at state 1). Recall that for a vector $v$, $\|v\| = \sqrt{\sum_{i=1}^n v_i^2} = \sqrt{v^Tv}$. Please graph $t$ on the $x$-axis and $\|Q^t\mu - \mathbf{1}/n\|$ on the $y$-axis. Let $n = 100$ in this case, and let $t$ go from 1 to 100.