18.085: Computational Science and Engineering
Problem Set 7 Solutions
Fall 2014
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Divide the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ into 4 unit squares (not triangles!) The values of u are given at the 8 boundary points (call those values u_E and u_{NE} and u_N ....). The only unknown is u_0 at the center. The one TEST function is BILINEAR in each square: $V(x, y) = a + bx + cy + dxy$. It is zero around the boundary and 1 at the center point. What are a b c d for its 4 pieces in the 4 squares? The one TRIAL function is also BILINEAR. It matches all 8 boundary values. We only have to find its value u_0 at the center, in terms of those 8 boundary values.

- Write the weak form of Laplace's equation.

- Substitute the 1 test function v and the trial function with unknown value at the center point.

- Integrate the weak form to find that unknown value in terms of the 8 known boundary values. This will be the difference equation for the BILINEAR choice of finite elements.

Solutions:

- We write the weak form of Laplace's equation as follows:

$$\iint \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dxdy = 0$$

- The boundaries are given below, with regions labeled 1, 2, 3, 4:
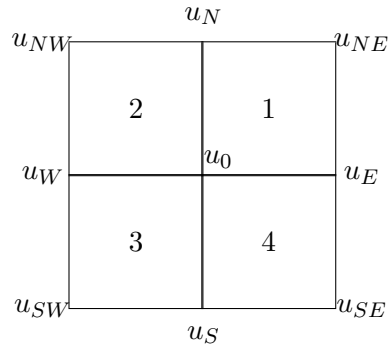
We now compute the bilinear test function V and trial function u for each region. One example computation will be provided for region 1, the other regions follow similar computation.

1: Observe $V(x, y) = a + bx + cy + dxy$. We need to solve for the coefficients.

$$
\begin{aligned}
V(0,0) = 1 &\implies a = 1 \\
V(1,0) = 0 = a + b &\implies b = -a = -1 \\
V(0,1) = 0 = a + c &\implies c = -1 \\
V(1,1) = 0 = a + b + c + d &\implies d = 1
\end{aligned}
$$

Hence, $V_1(x, y) = 1 - x - x + xy$. Now we move on to finding the trial function u.

$$u(0,0) = u_0 \implies a = u_0$$
$$u(1,0) = u_E = a + b \implies b = u_E - u_0$$
$$u(0,1) = u_N = a + c \implies c = u_N - u_0$$
$$u(1,1) = u_{NE} = a + b + c + d \implies d = u_{NE} + u_0 - u_N - u_E$$

Hence, $u_1(x, y) = u_0 + (u_E - u_0)x + (u_N - u_0)y + (u_{NE} + u_0 - u_N - u_E)xy$.

2: $V_2(x, y) = 1 + x - y - xy$, $u_2(x, y) = u_0 + (u_0 - u_W)x + (u_N - u_0)y + (u_W + u_N - u_0 - u_{NW})xy$

3: $V_3(x, y) = 1 + x + y + xy$, $u_3(x, y) = u_0 + (u_0 - u_W)x + (u_0 - u_S)y + (u_{SW} + u_0 - u_W - u_S)xy$

4: $V_4(x, y) = 1 - x + y - xy$, $u_4(x, y) = u_0 + (u_E - u_0)x + (u_0 - u_S)y + (u_E + u_S - u_0 - u_{SE})xy$

- We now compute the weak form integral. Again one example will be provided for region 1. The other regions follow similar computation. Many of you used MATLAB or other software to compute these tedious integrals, which is encouraged!

1: We compute the partials. If you used software to compute this, it is easiest to see that the weak form of Laplace's equation is really integrating a dot product of the gradients of V and u. If computing by hand, notice that there is a lot of symmetry here.

$$V_1(x, y) = 1 - x - y + xy \implies \nabla V_1 = (-1 + y, -1 + x)$$
$$u_1(x, y) = u_0 + (u_E - u_0)x + (u_N - u_0)y + (u_{NE} + u_0 - u_N - u_E)xy$$
$$\implies \nabla u_1 = ((u_e - u_0) + (u_{NE} - u_E - u_N + u_0)y, (u_N - u_0) + (u_{NE} - u_E - u_N + u_0)x)$$
$$\implies \nabla V_1 \cdot \nabla u_1 =$$
$$(-1 + y)((u_E - u_0) + (u_{NE} - u_e - u_N + u_0)y) + (-1 + x)((u_N - u_0) + (u_{NE} - u_E - u_N + u_0)x)$$

Now, integrating,

$$\iint \nabla V_1 \cdot \nabla u_1 dA =$$

$$\int_0^1 \int_0^1 (-1 + y)((u_E - u_0) + (u_{NE} - u_e - u_N + u_0)y) + (-1 + x)((u_N - u_0) + (u_{NE} - u_E - u_N + u_0)x) dA$$

where $dA = dxdy$ (done for space reasons). Integrating, we obtain that:

$$\int_0^1 \int_0^1 \nabla V_1 \cdot \nabla u_1 dxdy = \frac{1}{6}(4u_0 - u_N - u_E - 2u_{NE})$$

For the subsequent regions, either use symmetry or compute manually, remembering to modify the limits of integration for each region.

2:
$$\int_0^1 \int_{-1}^0 \nabla V_2 \cdot \nabla u_2 dxdy = \frac{1}{6}(4u_0 - u_N - u_W - 2u_{NW})$$

3:
$$\int_{-1}^{0}\int_{-1}^{0}\nabla V_3 \cdot \nabla u_3 dxdy = \frac{1}{6}(4u_0 - u_S - u_W - 2u_{SW})$$

4:
$$\int_{-1}^{0}\int_{0}^{1}\nabla V_4 \cdot \nabla u_4 dxdy = \frac{1}{6}(4u_0 - u_S - u_E - 2u_{SE})$$

To integrate over the entire square, we sum the four results above and equate to zero:

$$\frac{1}{6}(4u_0 - u_N - u_E - 2u_{NE}) + \frac{1}{6}(4u_0 - u_N - u_W - 2u_{NW}) + \frac{1}{6}(4u_0 - u_S - u_W - 2u_{SW}) + \frac{1}{6}(4u_0 - u_S - u_E - 2u_{SE}) = 0$$

$$\implies u_0 = \frac{1}{8}(u_N + u_E + u_S + u_W + u_{NE} + u_{NW} + u_{SE} + u_{SW}) \quad \square$$

In other words, u_0 is the average of the boundary values. From here, we can set up a difference equation for the bilinear choice of finite elements.