18.085 Homework 6

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1 3.3:1

$$\begin{cases} v_1 = \frac{\partial u}{\partial x} \\ v_2 = \frac{\partial u}{\partial y} \end{cases} \rightarrow \{ \ u(x,y) = x$$

$$\begin{cases} w_1 = \frac{\partial s}{\partial y} \\ w_2 = -\frac{\partial s}{\partial x} \end{cases} \rightarrow \{ \ s(x,y) = y$$

2 3.3:7

Using $w = (x^2, y^2)$.

$$div(w) = 2x + 2y$$

$$\begin{cases} w_1 = \frac{\partial s}{\partial y} \\ w_2 = -\frac{\partial s}{\partial x} \end{cases} \rightarrow div(w) = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} = \frac{\partial^2 s}{\partial x \partial y} - \frac{\partial^2 s}{\partial x \partial y} = 2x + 2y \neq 0$$

Using $w = (y^2, x^2)$.

$$\begin{cases} y^2 = \frac{\partial s}{\partial y} \\ x^2 = -\frac{\partial s}{\partial x} \end{cases} \rightarrow \begin{cases} s_1 = -\frac{1}{3}x^3 \\ s_2 = \frac{1}{3}y^3 \end{cases} \rightarrow s = -\frac{1}{3}x^3 + \frac{1}{3}y^3$$

3 3.3:8

$u = x^2$ then $w = div(u) = \left[\begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{array} \right]$. Show that:

$$\int \int_s \nabla \Delta u dx dy = \int_c n \cdot \nabla u ds$$

Focus on the LHS:

$$\Delta u = \left[\begin{array}{cc} 2x & 0 \end{array} \right]$$

$$\nabla \Delta u = 2$$

$$\int_{-1}^{1} \int_{-1}^{1} 2 dx dy = \int_{-1}^{1} 2x \Big|_{-1}^{1} dy = \int_{-1}^{1} 4 dy = 8$$

Focus on the RHS. By inspection the upper and lower sides of the square (parallel to the x axis) are normal to the direction of the potential lines, and will not experience any flow at all. We will calculate the left and right sides individually.

$$\int_C n \cdot \nabla u ds = \int_c n \cdot \begin{bmatrix} 2 & 0 \end{bmatrix} ds = \int_{c_1} \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 2x \\ 0 \end{bmatrix} ds + \int_{c_2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2x \\ 0 \end{bmatrix} ds$$

The two line integrals will be on the region of $t = [-1, 1]$ with the following parameterizations: $y = t; x = 1$ and $y = t; x = -1$.

Thus the final integrals will be:

$$\int_{-1}^{1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2*1 \\ 0 \end{bmatrix} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} + \int_{-1}^{1} \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 2*-1 \\ 0 \end{bmatrix} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} =$$
$$= \int_{-1}^{1} 2\sqrt{1+0} + \int_{-1}^{1} 2\sqrt{1+0} = 8$$

So we have proved the Divergence Theorem for this particular velocity function.

4 3.3:11

Using $v = (0, x)$ and $v = \nabla u = (u_x, u_y)$.

LHS:

$$\int\int \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) dxdy = \int\int (1-0) dxdy = xy + C$$

RHS:

$$\int v_1 dx + v_2 dy = \int xdy = xy + C$$

The two sides agree.

5 3.4:2

Using the boundary condition requirement itself solves the Poisson equation. If $x^2 + y^2 - 1 = 0$ we have $u_{xx} = 2$, and $u_{yy} = 2$, so $u_{xx} + u_{yy} = 4$.

6 3.4:4

We have given $u = r\cos\theta + \frac{\cos\theta}{r}$ and the Laplace's equation in cylindrical coordinates is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

$$\frac{\partial u}{\partial r} = \cos\theta - \frac{\cos\theta}{r^2}$$
$$\frac{\partial u}{\partial \theta} = -r\sin\theta - \frac{\sin\theta}{r}$$
$$\frac{\partial^2 u}{\partial r^2} = \frac{2\cos\theta}{r^3}$$
$$\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta - \frac{\cos\theta}{r}$$

Plugging everything in gives us:

$$\frac{2\cos\theta}{r^3} + \frac{1}{r}\left(\cos\theta - \frac{\cos\theta}{r^2}\right) + \frac{1}{r^2}\left(-r\cos\theta - \frac{\cos\theta}{r}\right) =$$
$$= \frac{2\cos\theta}{r^3} + \frac{\cos\theta}{r} - \frac{\cos\theta}{r^3} - \frac{\cos\theta}{r} - \frac{\cos\theta}{r^3} = 0$$

Thus the solution does indeed solve Laplace's equation. Using $r^2 = x^2 + y^2$ and $r\cos\theta = x$ we get $u(x, y)$:

$$u(x, y) = x + \frac{x}{x^2 + y^2}$$
$$u_x(x, y) = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$
$$u_y(x, y) = -\frac{2xy}{(x^2 + y^2)^2}$$

Using the vector field of $[x, y]$ under the constraint that $x^2 + y^2 = 1$, we have:

$$v \cdot n = \left[\begin{array}{cc} u_x & u_y \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] = x + \frac{xy^2 - x^3}{(x^2 + y^2)^2} - \frac{2xy^2}{(x^2 + y^2)^2} =$$
$$= x - xy^2 - x^3 = x(1 - y^2 - x^2) = x(1 - 1) = 0$$

This proves that our solution is normal to the unit circle.

7 3.4:5

Using $u = \log r$ and Laplace's equation in cylindrical coordinates: $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

$$\frac{\partial u}{\partial r} = \frac{1}{r}$$
$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}$$
$$-\frac{1}{r^2} + \frac{1}{r}\frac{1}{r} + 0 = 0$$

Laplace's equation holds for $u = \log r$.

If we use a $z = re^{i\theta}$ and we take the log, we get $\log z = \log r + i\theta$. Since $re^{i\theta}$ solves Laplace's equation, so does $\log z$, therefore $s = \theta$ for this case.

Using $U = \log r^2$ we get:

$$\frac{\partial U}{\partial r} = \frac{1}{r^2}2r = \frac{2}{r}$$
$$\frac{\partial^2 U}{\partial r^2} = -\frac{2}{r^2}$$
$$-\frac{2}{r^2} + \frac{1}{r}\frac{2}{r} + 0 = 0$$

Laplace's equation holds for $U = \log r^2$.

Using a $Z = z^2 = r^2 e^{2i\theta}$ taking the log we get: $\log Z = \log r^2 + 2i\theta$. Therefore the $S = 2\theta$.

8 Extra

a. What is $v(x, y) = \nabla u$?

$$u(x, y) = x + y - x^2 + y^2$$
$$v(x, y) = \nabla u = \begin{bmatrix} {}^{\partial u}/_{\partial x} \\ {}^{\partial u}/_{\partial y} \end{bmatrix} u(x, y) = \begin{bmatrix} 1 - 2x \\ 1 + 2y \end{bmatrix}$$

b. What is the stream function $s(x, y)$?

$$\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} = 1 - 2x$$
$$\frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x} = 2y + 1$$
$$s_1 = y - 2xy + f(x)$$
$$s_2 = -2yx - x + f(y)$$

Therefore $s(x, y) = -x + y - 2xy$.

c. Find $g = \nabla s$

$$g = \nabla s(x,y) = \begin{bmatrix} {}^{\partial s}\!/\!{}_{\partial x} \\ {}^{\partial s}\!/\!{}_{\partial y} \end{bmatrix} s(x,y) = \begin{bmatrix} -2y-1 \\ -2x+1 \end{bmatrix}$$

Verifying that $v \perp g$:

$$\begin{aligned} v \cdot g^T = & \begin{bmatrix} 1-2x \\ 1+2y \end{bmatrix} \begin{bmatrix} -2y-1 & -2x+1 \end{bmatrix} \\ = & (1-2x)(-2y-1) + (1+2y)(-2x+1) \\ = & -2y - 1 + 4xy + 2x - 2x + 1 - 4xy + 2y = 0 \end{aligned}$$

Since the dot product is zero, the two must be orthogonal.

d. Identify curve types

Both types are hyperbolas, just of different kinds. $x^2 - y^2 + 2xy = c$ is one type of hyperbola, and $y = \frac{C}{-2x+1} + \frac{x}{-2x+1}$ is another type.