#63 Rachel Mok
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18.085 PS 5, problem 1: pg 1

$$u'' + u = x \qquad \text{B.Cs:} \quad u(0) = 0$$
$$u(1) = 0$$

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Solve with $u = u_{part.} + u_{null}$:

$$u = x + A\sin x + B\cos x$$

Apply BCs:

$$u(0) = 0 \Rightarrow 0 = 0 + A\sin(0) + B\cos(0)$$

$$0 = B$$

$$u(1) = 0 \Rightarrow 0 = 1 + A\sin(1) + \cancel{B}\cos(1)$$

$$A = \frac{-1}{\sin(1)}$$

exact soln:

$$\boxed{u(x) = x - \frac{\sin(x)}{\sin(1)}}$$

Finding weak form:

$$\int_0^1 (u'' + u)\, v(x)\, dx = \int_0^1 x\, v(x)\, dx$$

$$\int_0^1 \frac{d}{dx}\left(\frac{du}{dx}\right) v(x)\, dx + \int_0^1 u(x)\, v(x)\, dx = \int_0^1 x\, v(x)\, dx$$

$$\int_0^1 \frac{du}{dx}\left(-\frac{dv}{dx}\right) dx + \underbrace{\left[\frac{du}{dx} v(x)\right]_0^1}_{=0 \text{ if } v(0)=0,\ v(1)=0} + \int_0^1 u(x)\, v(x)\, dx = \int_0^1 x\, v(x)\, dx$$

$$\Rightarrow \int_0^1 \frac{du}{dx}\left(-\frac{dv}{dx}\right) dx + \int_0^1 u(x)\, v(x)\, dx = \int_0^1 x\, v(x)\, dx$$

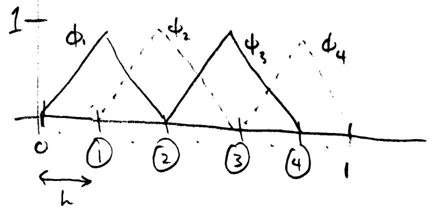
Using N hat functions,

$$\int_0^1 \frac{dU}{dx}\left(-\frac{dV_i}{dx}\right)dx + \int_0^1 U(x)\,V_i(x)\,dx = \int_0^1 x\,V_i(x)\,dx \qquad \text{for } i=1,2,\dots,N$$

$$U(x) = u_1\phi_1(x) + u_2\phi_2(x) + \dots + u_N\phi_N(x)$$

For $N=4$:

$\phi_i(x) = V_i(x)$



$(N+1)h = 1$

$h = \frac{1}{5}$

$f(x) = x$

$$\phi_1 = \begin{cases} \frac{x}{h} & 0<x<h \\ -\frac{x}{h}+2 & h<x<2h \\ 0 & x>2h \end{cases} \qquad \phi_2 = \begin{cases} \frac{x}{h}-1 & h<x<2h \\ -\frac{x}{h}+3 & 2h<x<3h \\ 0 & x<h \text{ \& } x>3h \end{cases}$$

$$F_i = \int_0^1 x V_i(x)\,dx \Rightarrow F_1 = \int_0^1 x V_1\,dx = \int_0^h x\left(\frac{x}{h}\right)dx + \int_h^{2h} x\left(-\frac{x}{h}+2\right)dx + \int_{2h}^1 0\,dx$$

$$\Rightarrow \boxed{F = h\begin{bmatrix} h \\ 2h \\ \vdots \\ Nh \end{bmatrix}_{N\times 1}}$$

$$F_2 = \int_0^1 x V_2\,dx = \int_0^h 0\,dx + \int_h^{2h} x\left(\frac{x}{h}-1\right)dx + \int_{2h}^{3h} x\left(-\frac{x}{h}+3\right)dx + \int_{3h}^1 0\,dx$$

$$= \frac{5h^2}{6} + \frac{7h^2}{6} = 2h^2$$

$$\Rightarrow F_1 = \frac{1}{h}\frac{h^3}{3} + \left[-\frac{1}{h}\frac{x^3}{3} + 2\frac{x^2}{2}\right]_h^{2h}$$

$$= \frac{h^2}{3} - \frac{1}{3h}\left(8h^3 - h^3\right) + \left(4h^2 - h^2\right)$$

$$= \frac{h^2}{3} - \frac{7h^2}{3} + 3h^2$$

$$= h^2$$

For K:

$$\int_0^1 \left[u_1\phi_1'(x) + u_2\phi_2'(x) + \dots u_N\phi_N'(x)\right]\left(-V_i'\right)dx$$

$$K_{11} = \int_0^1 \phi_1'(x)\left(-V_1'(x)\right)dx = \int_0^h \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx + \int_h^{2h}\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dx = -\frac{2}{h}$$

$$\phi_1' = V_1' = \begin{cases} \frac{1}{h} & 0<x<h \\ -\frac{1}{h} & h<x<2h \\ 0 & x>2h \end{cases}$$

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18.085 PS5, problem 1: pg 2

$$K_{12} = \int_0^1 \phi_1'(x)\,(-V_2'(x))\,dx$$

$$\phi_1'(x) = \begin{cases} \frac{1}{h} & 0 < x < h \\ -\frac{1}{h} & h < x < 2h \\ 0 & x > 2h \end{cases}$$

$$\phi_2'(x) = V_2'(x) = \begin{cases} \frac{1}{h} & h < x < 2h \\ -\frac{1}{h} & 2h < x < 3h \\ 0 & x > 3h \;\&\; x < h \end{cases}$$

$$K_{12} = \int_0^h \left(\tfrac{1}{h}\right)(0)\,dx + \int_h^{2h} \left(-\tfrac{1}{h}\right)\left(-\tfrac{1}{h}\right)dx + \int_{2h}^{3h} (0)\left(+\tfrac{1}{h}\right)dx + \int_{3h}^{1} 0\,dx$$

$$= \frac{1}{h}$$

Thus,

$$\boxed{K = \frac{1}{h}\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix}_{N \times N}}$$

For M:

$$\int_0^1 \left[u_i \phi_1(x) + u_2 \phi_2(x) + \cdots u_N \phi_N(x) \right] V_i(x)\, dx$$

$$M_{11} = \int_0^1 \phi_1(x) V_1(x)\, dx = \int_0^1 (\phi_1(x))^2\, dx = \int_0^h \left(\frac{x}{h}\right)^2 dx + \int_h^{2h} \left(-\frac{x}{h}+2\right)^2 dx + \int_{2h}^1 0\, dx$$

$$\phi_1(x) = \begin{cases} \frac{x}{h} & 0 < x < h \\ -\frac{x}{h} + 2 & h < x < 2h \\ 0 & x > 2h \end{cases} \qquad \phi_2(x) = \begin{cases} \frac{x}{h} - 1 & h < x < 2h \\ -\frac{x}{h} + 3 & 2h < x < 3h \\ 0 & x < h \text{ \& } x > 3h \end{cases}$$

$$M_{11} = \frac{1}{h^2}\left(\frac{h^3}{3} - 0\right) + \int_h^{2h} \frac{x^2}{h^2} - \frac{4x}{h} + 4\, dx$$

$$= \frac{h}{3} + \left[\frac{1}{h^2}\frac{x^3}{3} - \frac{4}{h}\frac{x^2}{2} + 4x\right]_h^{2h}$$

$$= \frac{h}{3} + \left[\frac{1}{3h^2}(8h^3 - h^3) - \frac{2}{h}(4h^2 - h^2) + 4(2h - h)\right]$$

$$= \frac{h}{3} + \left[\frac{1}{3h^2}(7h^3) - \frac{2}{h}(3h^2) + 4(h)\right]$$

$\frac{7}{3} - 6 + 4$

$$= \frac{2}{3}h$$

$\frac{7}{3} - \frac{6}{3} = \frac{1}{3}$

$$M_{12} = \int_0^1 \phi_1(x) V_2(x)\, dx = \int_0^h \frac{x}{h}(0)\, dx + \int_h^{2h} \left(-\frac{x}{h}+2\right)\left(\frac{x}{h}-1\right) dx + \int_{2h}^{3h} (0)\left(-\frac{x}{h}+3\right) dx + \int_{3h}^1 0\, dx$$

$$= \int_h^{2h} \left[-\frac{x^2}{h^2} + \frac{x}{h} + \frac{2x}{h} - 2\right] dx$$

$$= \frac{h}{6}$$

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$$\boxed{M = h \begin{bmatrix} 2/3 & \frac{1}{6} & & 0 \\ \frac{1}{6} & 2/3 & \ddots & \\ & \ddots & \ddots & 1/6 \\ 0 & & \frac{1}{6} & 2/3 \end{bmatrix}_{N \times N}}$$

✓

Now can solve:

$$(K + M)U = F \Rightarrow \left(\frac{1}{h} \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} + h \begin{bmatrix} 2/3 & 1/6 & & \\ \frac{1}{6} & 2/3 & \ddots & \\ & \ddots & \ddots & 1/6 \\ & & 1/6 & 2/3 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = h \begin{bmatrix} h \\ 2h \\ 3h \\ \vdots \\ Nh \end{bmatrix}$$

Looking for:

$$\text{max error} \propto \frac{1}{N^2}$$

$$\log(\text{max error}) \propto \log\left(\frac{1}{N^2}\right)$$

$$\log(\text{max error}) \propto -2 \log(N)$$

Thus the slope of a log (max error) vs. log (N) plot should be close to -2.
From MATLAB, I get a slope of -1.8125

#63

```
%18.085: PSet 5 problem 1

clc
clear
close all

N_vec=[4,8,16,32];
n=length(N_vec);

A=-1/sin(1);
B=0;

max_error=NaN(n,1);

approx_sol=cell(n,2);

for i=1:n
    N=N_vec(i);
    h=1/(N+1);

    x_vec=(h:h:N*h)';
    u_ex=A*sin(x_vec)+B*cos(x_vec)+x_vec;

    K=1/(h^2)*toeplitz([-2 1 zeros(1,N-2)]);
    M=toeplitz([2/3, 1/6, zeros(1,N-2)]);
    F=(h:h:N*h)';

    u_approx=(K+M)\F;

    u_error=abs(u_ex-u_approx);
    max_error(i)=max(u_error);

    approx_sol{i,1}=x_vec;
    approx_sol{i,2}=u_approx;
end

%apply BCs to exact solution:
x_vec=vertcat(0,x_vec,1);
u_ex=vertcat(0,u_ex,0);

%creating figure:
x_vec_approx=approx_sol{n,1};
u_approx=approx_sol{n,2};
line1=sprintf('+d^2u/dx^2+u=x for N=%5d',N);

figure
hold on
plot(x_vec_approx,u_approx,'og','Markerfacecolor','g')
plot(x_vec,u_ex,'b-')
hold off
title(line1)
xlabel('x')
```

#63

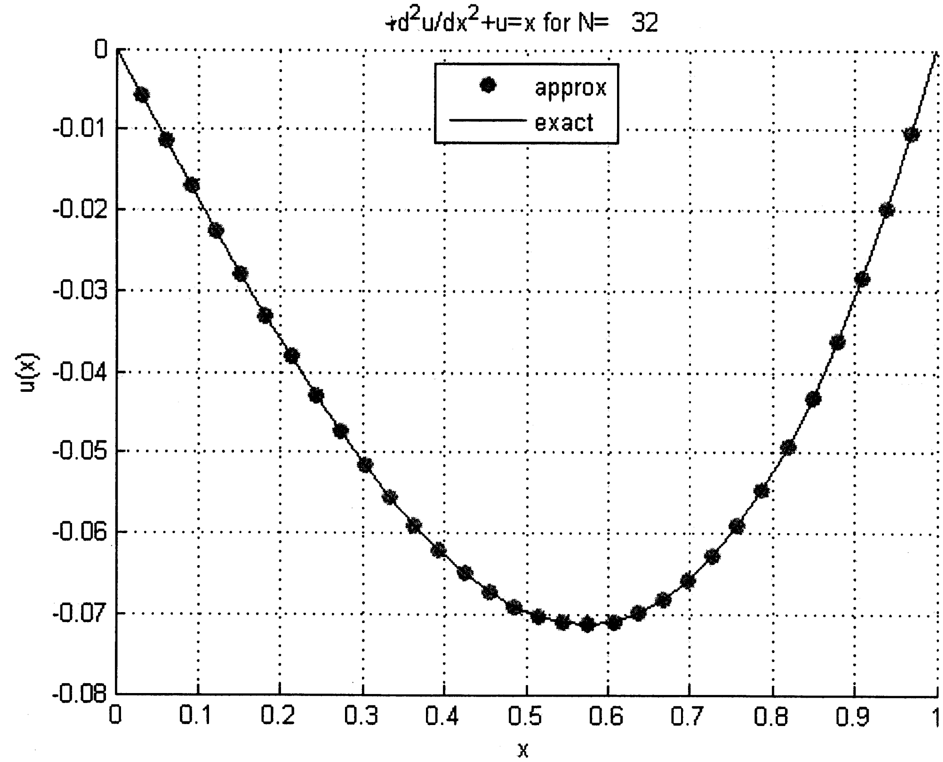
```
ylabel('u(x)')
legend('approx','exact',0)
grid on
axis on

%least squares to find best fitting line:
log_N=log(N_vec');
log_max_error=log(max_error);

A_mat=[ones(n,1),log_N];
b=log_max_error;

u_hat=(A_mat'*A_mat)\(A_mat'*b);
pwr_N=u_hat(2)
least_sqrs_line=u_hat(1)*ones(n,1)+u_hat(2)*log_N;

%creating error plot
figure
hold on
plot(log_N,least_sqrs_line,'b')
plot(log_N,log_max_error,'og','Markerfacecolor','g')
hold off
title('Log log plot of max error vs N')
xlabel('Log(N)')
ylabel('Log(max error)')
legend('best fit line')
grid on
axis on
```

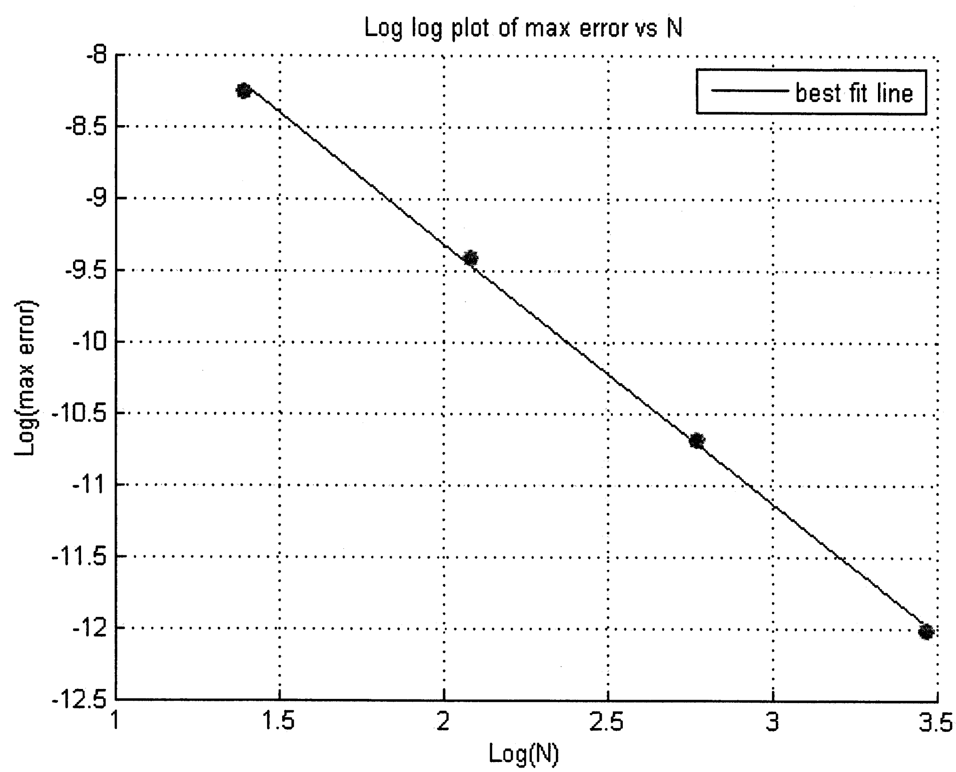
pwr_N =

-1.8125

√

#63


+d2u/dx2+u=x for N= 32
approx
exact
u(x)
x


Log log plot of max error vs N
best fit line
Log(max error)
Log(N)

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18.085 PS 5, problem 3.1.5 : pg 1

$$C = \begin{cases} c_1 = 1 & x \le \frac{1}{2} \\ c_2 = 2 & x > \frac{1}{2} \end{cases} \quad ; \quad f = \text{const.}$$

$$-\frac{dw}{dx} = f \qquad \text{B.C.}: \; w(1) = 0$$

$$w(x) = -fx + A$$

Apply BC:

$$0 = -f(1) + A$$

$$A = f$$

$$\boxed{w(x) = f[1-x]}$$

✓

$$c\frac{du}{dx} = w \qquad \text{B.C.} \; u(0) = 0$$

$$c\frac{du}{dx} = f[1-x]$$

For $0 \le x \le \frac{1}{2}$:

$$\frac{du}{dx} = \frac{f}{c_1}[1-x]$$

$$u(x) = \frac{f}{c_1}\left(x - \frac{x^2}{2}\right) + B$$

Apply B.C: $u(0) = 0$

$$0 = B$$

At $x = \frac{1}{2}$: $u(\frac{1}{2}) = \frac{f}{c_1}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2\right] = \frac{f}{c_1}\left(\frac{3}{8}\right)$

For $\frac{1}{2} < x \le 1$

$$\frac{du}{dx} = \frac{f}{c_2}[1-x]$$

$$u(x) = \frac{f}{c_2}\left[x - \frac{x^2}{2}\right] + D$$

Apply BC.: $u(\frac{1}{2}) = \frac{3}{8}\frac{f}{c_1}$

$$\frac{3}{8}\frac{f}{c_1} = \frac{f}{c_2}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2\right] + D \Rightarrow D = \frac{3}{8}f\left(\frac{1}{c_1} - \frac{1}{c_2}\right)$$

$$\Rightarrow \boxed{u(x) = \begin{cases} f\left(x - \frac{x^2}{2}\right) & 0 \le x \le \frac{1}{2} \\ \frac{f}{2}\left(x - \frac{x^2}{2}\right) + \frac{3}{16}f & \frac{1}{2} < x \le 1 \end{cases}}$$

✓

18.085 PS 5, problem 3.1.9 : pg 1

$$-cu'' + u' = 1 \quad w/ \text{ B.C.s: } u(0) = u(1) = 0$$

Soln: $u(x) = d_1 + d_2 e^{x/c} + x$

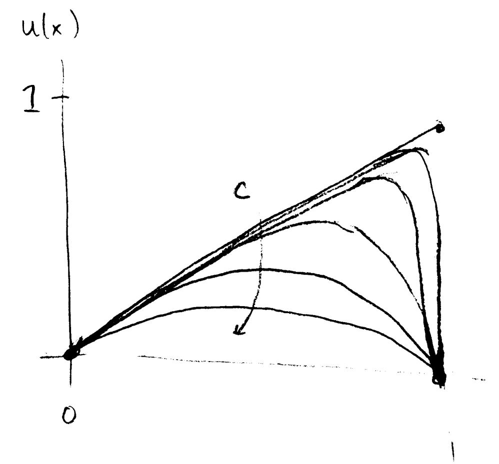
Apply BCs:

$$0 = d_1 + d_2 e^{(0)/c} + (0) \Rightarrow d_1 = -d_2$$

$$0 = d_1 + d_2 e^{1/c} + 1$$

$$\Rightarrow 0 = -d_2 + d_2 e^{1/c} + 1$$

$$-1 = d_2 \left[e^{1/c} - 1 \right]$$



$$\boxed{d_2 = \frac{-1}{e^{1/c} - 1} \; ; \; d_1 = \frac{1}{e^{1/c} - 1} \qquad \lim_{c \to 0} d_2 = 0 \qquad \lim_{c \to 0} d_1 = 0}$$

√

$\lim_{c \to 0} u(x) = x$ which does satisfy $U' = 1$

For $\lim_{c \to 0} u(x) = U(x) = x$, the boundary values are:

$$U(0) = 0$$

$$U(1) = 1$$

So the $u(0) = 0$ is fulfilled, but not the $u(1) = 0$. There is a boundary layer at $x = 1$.

#63 Rachel Mak
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18.085 PS 5, problem 3.1.11 : pg 1

$-u'' = x \quad w/ \quad u(0) = u(1) = 0$.

$$\frac{d^2u}{dx^2} = -x$$

$$\frac{du}{dx} = -\frac{x^2}{2} + C_1$$

$$u(x) = -\frac{1}{2}\frac{x^3}{3} + C_1 x + C_2$$

Apply BCs:

$$0 = -\frac{1}{6}(0)^3 + C_1(0) + C_2 \Rightarrow C_2 = 0$$

$$0 = -\frac{1}{6}(1)^3 + C_1(1) + C_2$$

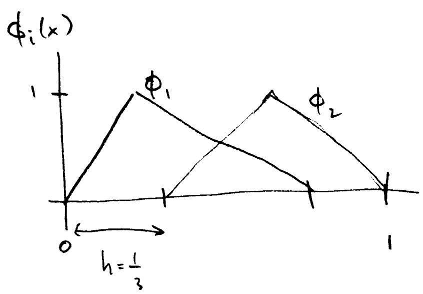
$$C_1 = \frac{1}{6}$$

$$\boxed{u(x) = -\frac{1}{6}x^3 + \frac{1}{6}x}$$

✓

Weak form: $\displaystyle\int_0^1 -\frac{d}{dx}\left(\frac{du}{dx}\right) V(x)\,dx = \int_0^1 x\,V(x)\,dx$

$$\int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx + \underbrace{\left[-\frac{du}{dx}v(x)\right]_0^1}_{=0 \text{ if } v(0)=v(1)=0} = \int_0^1 x\,V(x)\,dx$$



$U(x) = u_1\phi_1(x) + u_2\phi_2(x)$ $\quad \Rightarrow \quad \displaystyle\int_0^1 \frac{du}{dx}\frac{dV}{dx}\,dx = \int_0^1 x\,V\,dx$

Discrete Version:

$$\int_0^1 \left[u_1\phi_1' + u_2\phi_2'\right]\frac{dV_i}{dx}\,dx = \int_0^1 x\,V_i\,dx \qquad i = 1, 2$$

$$\phi_1 = \begin{cases} \frac{x}{h} & 0 < x < h \\ -\frac{x}{h} + 2 & h < x < 2h \\ 0 & x > 2h \end{cases} \qquad \phi_2 = \begin{cases} \frac{x}{h} - 1 & h < x < 2h \\ -\frac{x}{h} + 3 & 2h < x < 3h \\ 0 & x < h \ \& \ x > 3h \end{cases}$$

$$F_1 = \int_0^1 x V_1(x)\,dx = \int_0^h x\left(\frac{x}{h}\right)dx + \int_h^{2h} x\left(-\frac{x}{h}+2\right)dx + \int_{2h}^1 0\,dx$$

$$= \frac{h^2}{3} + \frac{2h^2}{3}$$

$$= h^2$$

$$F_2 = \int_0^1 x\,V_2(x)\,dx = \int_0^h 0\,dx + \int_h^{2h} x\left(\frac{x}{h} - 1\right)dx + \int_{2h}^{3h} x\left(-\frac{x}{h}+3\right)dx + \int_{3h}^1 0\,dx$$

$$= \frac{5h^2}{6} + \frac{7h^2}{6}$$

$$= 2h^2$$

$$F = \begin{bmatrix} h^2 \\ 2h^2 \end{bmatrix}$$

$$K_{11} = \int_0^1 \phi_1' V_1'\,dx = \int_0^h \left(\frac{1}{h}\right)\left(\frac{1}{h}\right)dx + \int_h^{2h}\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx + \int_{2h}^1 0\,dx$$

$$= \frac{2}{h}$$

$$K_{12} = \int_0^1 \phi_1' V_2'\,dx = \int_0^h \left(\frac{1}{h}\right)(0)\,dx + \int_h^{2h}\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dx + \int_{2h}^{3h}(0)\left(-\frac{1}{h}\right)dx + \int_{3h}^1 0\,dx$$

$$= -\frac{1}{h}$$

#63 R.M.

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18.085 PS 5, problem 3.1.11 : pg 2

$$\frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = h^2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$2u_1 - u_2 = h^3$$

$h = \frac{1}{3}$

$$(+)\ 2\left[-u_1 + 2u_2 = 2h^3\right]$$

$$3u_2 = 5h^3$$

$$u_2 = \frac{5}{3}h^3 = \frac{5}{81}$$

$$u_1 = 2u_2 - 2h^3$$

$$u_1 = \frac{4}{81}$$

$$u_{approx}: \begin{cases} u - 0 = \left(\frac{\frac{4}{81} - 0}{\frac{1}{3} - 0}\right)(x - 0) & [0, \frac{1}{3}] \\ u - \frac{4}{81} = \left(\frac{\frac{5}{81} - \frac{4}{81}}{\frac{2}{3} - \frac{1}{3}}\right)(x - \frac{1}{3}) & (\frac{1}{3}, \frac{2}{3}] \\ u - 0 = \left(\frac{0 - \frac{5}{81}}{1 - \frac{2}{3}}\right)(x - 1) & (\frac{2}{3}, 1] \end{cases}$$

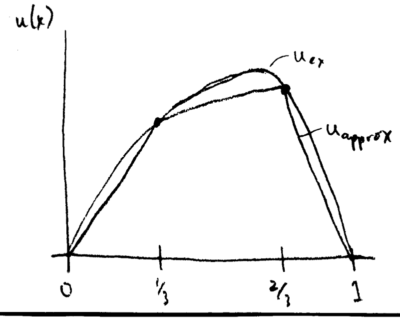
$$\boxed{u_{approx} = \begin{bmatrix} \frac{4}{81} \\ \frac{5}{81} \end{bmatrix}}$$

$$u_{exact} = \begin{bmatrix} \frac{1}{6}\left[-\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)\right] \\ \frac{1}{6}\left[-\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)\right] \end{bmatrix} = \begin{bmatrix} 4/81 \\ 5/81 \end{bmatrix}$$

✓

$$u(x) = \frac{1}{6}\left[-x^3 + x\right]$$

The values u_1 & u_2 agree exactly with at the mesh points.



$$u_{approx}: \begin{cases} u = \frac{4}{27}x & 0 \le x \le \frac{1}{3} \\ u = \frac{1}{27}x + \frac{1}{27} & \frac{1}{3} < x \le \frac{2}{3} \\ u = -\frac{5}{27}x + \frac{5}{27} & \frac{2}{3} < x \le 1 \end{cases}$$

error:

$$u_{error} = \left| u_{ex} - u_{approx} \right|$$

$$= \begin{cases} -\frac{1}{6}x^3 + \frac{1}{6}x - \frac{4}{27}x & [0, \frac{1}{3}] \\ -\frac{1}{6}x^3 + \frac{1}{6}x - \left(\frac{1}{27}x + \frac{1}{27}\right) & (\frac{1}{3}, \frac{2}{3}] \\ -\left(-\frac{1}{6}x^3 + \frac{1}{6}x\right) + \left(-\frac{5}{27}x + \frac{5}{27}\right) & (\frac{2}{3}, 1] \end{cases}$$

$$= \begin{cases} -\frac{1}{6}x^3 + \frac{1}{54}x & [0, \frac{1}{3}] \\ -\frac{1}{6}x^3 + \frac{7}{54}x - \frac{1}{27} & (\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{6}x^3 - \frac{19}{54}x + \frac{5}{27} & (\frac{2}{3}, 1] \end{cases}$$

To find max error, take derivative:

$\frac{du_{error}}{dx} =$	$\frac{du_{error}}{dx} = 0$ at:	$u_{error}(x_{max})$	
$-\frac{1}{2}x^2 + \frac{1}{54}$	$x_{max} = \frac{\sqrt{3}}{9}$	0.0024	$[0, \frac{1}{3}]$
$-\frac{1}{2}x^2 + \frac{7}{54}$	$x_{max} = \frac{\sqrt{21}}{9}$	0.0070	$(\frac{1}{3}, \frac{2}{3}]$
$\frac{1}{2}x^2 - \frac{19}{54}$	$x_{max} = \frac{\sqrt{57}}{9}$	0.0116	$(\frac{2}{3}, 1]$

$$\boxed{\text{Largest error occurs at } x = \frac{\sqrt{57}}{9}}$$

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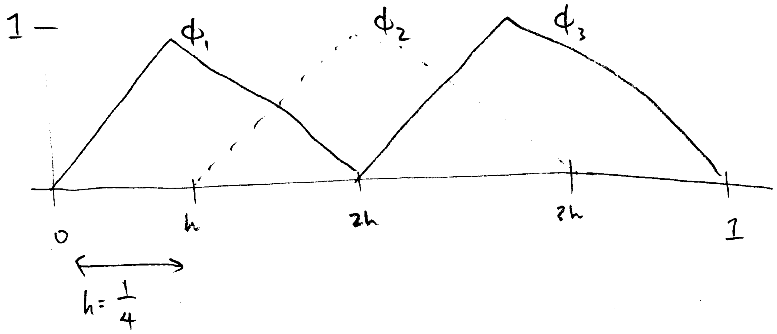


$$M_{ij} = \int V_i V_j \, dx$$

$$\phi_1 = \begin{cases} \frac{x}{h} & 0 < x < h \\ -\frac{x}{h} + 2 & h < x < 2h \\ 0 & x > 2h \end{cases} \qquad \phi_2 = \begin{cases} \frac{x}{h} - 1 & h < x < 2h \\ -\frac{x}{h} + 3 & 2h < x < 3h \\ 0 & x < h \ \& \ x > 3h \end{cases} \qquad \phi_3 = \begin{cases} \frac{x}{h} - 2 & 2h < x < 3h \\ -\frac{x}{h} + 4 & 3h < x < 4h \\ 0 & x < 2h \end{cases}$$

$$M_{11} = \int_0^1 V_1(x) V_1(x) \, dx = \int_0^1 [V_1(x)]^2 \, dx$$

$$= \int_0^h \left(\frac{x}{h}\right)^2 dx + \int_h^{2h} \left(-\frac{x}{h} + 2\right)^2 dx + \int_{2h}^1 0 \, dx$$

$$= \frac{1}{h^2}\left[\frac{h^3}{3} - 0\right] + \int_h^{2h} \frac{x^2}{h^2} - \frac{4x}{h} + 4 \, dx$$

$$= \frac{h}{3} + \left[\frac{1}{h^2}\frac{x^3}{3} - \frac{4}{h}\frac{x^2}{2} + 4x\right]_h^{2h}$$

$$= \frac{h}{3} + \left[\frac{1}{3h^2}(8h^3 - h^3) - \frac{2}{h}(4h^2 - h^2) + 4(2h - h)\right]$$

$$= \frac{2}{3}h$$

✓

$$M_{12} = \int_0^1 V_1(x)\, V_2(x)\, dx$$

$$= \int_0^h \frac{x}{h}(0)\, dx + \int_h^{2h} \left(-\frac{x}{h}+2\right)\left(\frac{x}{h}-1\right) dx + \int_{2h}^{3h} (0)\left(-\frac{x}{h}+3\right) dx + \int_{3h}^{1} 0\, dx$$

$$= \int_h^{2h} \left[-\frac{x^2}{h^2} + \frac{x}{h} + \frac{2x}{h} - 2\right] dx$$

$$= \frac{h}{6}$$

$$M_{13} = \int_0^1 V_1(x)\, V_3(x)\, dx$$

$$= 0 \quad \text{(no overlap)}$$

Thus :

$$\boxed{M = h \begin{bmatrix} 2/3 & \frac{1}{6} & 0 \\ \frac{1}{6} & 2/3 & \frac{1}{6} \\ 0 & \frac{1}{6} & 2/3 \end{bmatrix}}$$

$h = \frac{1}{4}$

18.085 PS5, 3.1.13 pg 1

#13 Rachel Nuki

$$\frac{d}{dx}\left(w(x)\,v(x)\right) = \frac{dw}{dx}v(x) + w(x)\frac{dv}{dx}$$

$$\int_0^1 \frac{d}{dx}\left(w(x)v(x)\right)dx = \int_0^1 \left[\frac{dw}{dx}v(x) + w(x)\frac{dv}{dx}\right]dx$$

$$w(x)v(x)\Big|_0^1 = \int_0^1 \frac{dw}{dx}v(x)\,dx + \int_0^1 w(x)\frac{dv}{dx}\,dx$$

$$\boxed{\int_0^1 \frac{dw}{dx}v\,dx = \left[wv\right]_0^1 - \int_0^1 w\frac{dv}{dx}\,dx}$$

If $w = c\dfrac{du}{dx}$:

$$\int_0^1 \frac{d}{dx}\left(c\frac{du}{dx}\right)v\,dx = \left[c\frac{du}{dx}v\right]_0^1 - \int_0^1 c\frac{du}{dx}\frac{dv}{dx}\,dx$$

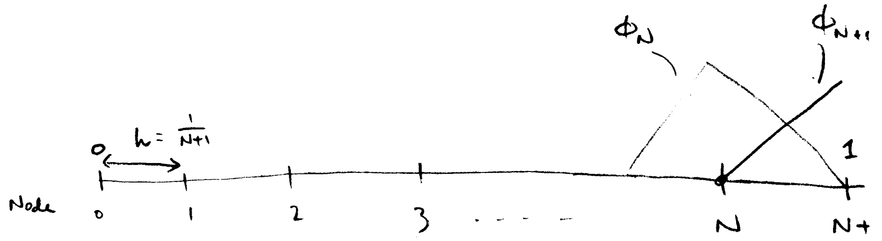
Multiply by -1:

$$\boxed{\int_0^1 -\frac{d}{dx}\left(c\frac{du}{dx}\right)v\,dx = \int_0^1 c\frac{du}{dx}\frac{dv}{dx}\,dx - \left[c\frac{du}{dx}v\right]_{x=0}^{x=1}}$$

✓

#63 Rachel Mde
10/15/14

18.085 PS5, problem 3.1.18: pg 1



$-u_{xx} = f(x)$

BC. $u(0) = 0$

$u'(1) = 0$

(a) Last row of K_{N+1}:

$$K_{N+1,N} = \int_0^1 c(x) \frac{d\phi_{N+1}}{dx} \frac{d\phi_N}{dx}\, dx$$

$$= \int_{Nh}^{(N+1)h} (1)\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx$$

$$= -\frac{h}{h^2} = -\frac{1}{h}$$

$$K_{N+1,N+1} = \int_0^1 c(x) \frac{d\phi_{N+1}}{dx} \frac{d\phi_{N+1}}{dx}\, dx$$

$$= \int_{Nh}^{(N+1)h} (1)\left(\frac{1}{h}\right)\left(\frac{1}{h}\right) dx$$

✓

$$= \frac{h}{h^2} = \frac{1}{h}$$

$$\boxed{K_{N+1} = \frac{1}{h}\left[\underbrace{0 \;\cdots\; 0}_{(N+1)-2 \text{ zeros}} \;\; -1 \;\; 1\right]}$$

$\boxed{u'(1) = 0 \text{ is represented as } \frac{1}{h}(u_{N+1} - u_N)}$

(b) $f_0 = \text{const.}$

$$F_{N+1} = \int_0^1 f_0 \, V_{N+1} \, dx$$

$$= f_0 \frac{1}{2}\left[h\right](1)$$

$$\boxed{F_{N+1} = f_0 \frac{h}{2}}$$

Solve $K_{N+1} U = F$:

Using $N = 2$, $h = \frac{1}{3}$

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = f_0 h \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$$

$$(3)^2\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = f_0 \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$$

$$\boxed{U = \begin{bmatrix} 5/18 \\ 4/9 \\ 1/2 \end{bmatrix} f_0}$$

Compare to actual values:

$$U_{ex} = f_0 \begin{bmatrix} \left(\frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{3}\right)^2 \\ \left(\frac{2}{3}\right) - \frac{1}{2}\left(\frac{2}{3}\right)^2 \\ (1) - \frac{1}{2}(1)^2 \end{bmatrix} = f_0 \begin{bmatrix} 5/18 \\ 4/9 \\ 1 \end{bmatrix}$$

$\boxed{\text{Values at mesh points match exactly with true values!}}$

#63

```
%18.085 PS5 problem 3.1.18

clc
clear
close all

format rational

h=1/3
K=(1/h)*[2,-1,0;-1,2,-1;0,-1,1]
F=h*[1;1;1/2]

U=K\F
```

```
h =

       1/3


K =

       6              -3               0
      -3               6              -3
       0              -3               3


F =

       1/3
       1/3
       1/6


U =

       5/18
       4/9
       1/2
```

✓

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#63 Rachel Mc
10/16/14

18.085 PS 5, problem 3.2.3 : pg. 1

$u'''' = \delta(x)$

B.C.

At $x = -1$, $u = 0$, $u' = 0$

At $x = 1$, $u = 0$, $u' = 0$

Eqn 1:

$$u(x) = \begin{cases} A + Bx + Cx^2 + Dx^3 - \frac{1}{6}x^3 & x \le 0 \\ A + Bx + Cx^2 + Dx^3 & x \ge 0 \end{cases}$$

$D = d + \frac{1}{6}$

$d = D - \frac{1}{6}$

$$u'(x) = \begin{cases} B + 2Cx + 3Dx^2 - \frac{1}{2}x^2 & x \le 0 \\ B + 2Cx + 3Dx^2 & x \ge 0 \end{cases}$$

Apply BCs:

$u(-1) = A - B + C - D + \frac{1}{6} = 0$ ✓

$u'(-1) = B - 2C + 3D - \frac{1}{2} = 0$

$u(1) = A + B + C + D = 0$

$u'(1) = B + 2C + 3D = 0$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1/24 \\ 0 \\ -1/8 \\ \frac{1}{12} \end{bmatrix}}$$

$$\Rightarrow \boxed{u(x) = \begin{cases} \frac{1}{24} - \frac{1}{8}x^2 - \frac{1}{12}x^3 & x \leq 0 \\ \frac{1}{24} - \frac{1}{8}x^2 + \frac{1}{12}x^3 & x \geq 0 \end{cases}}$$

✓

#63

```
%18.085 PS5 problem 3.2.3

clc
clear
close all

format rational

A=[1,-1,1,-1;0,1,-2,3;1,1,1,1;0,1,2,3]
b=[-1/6;1/2;0;0]

x=A\b
```

```
A =

     1            -1             1            -1
     0             1            -2             3
     1             1             1             1
     0             1             2             3


b =

    -1/6
     1/2
     0
     0


x =

     1/24
    -1/72057594037927936
    -1/8
     1/12
```

✓

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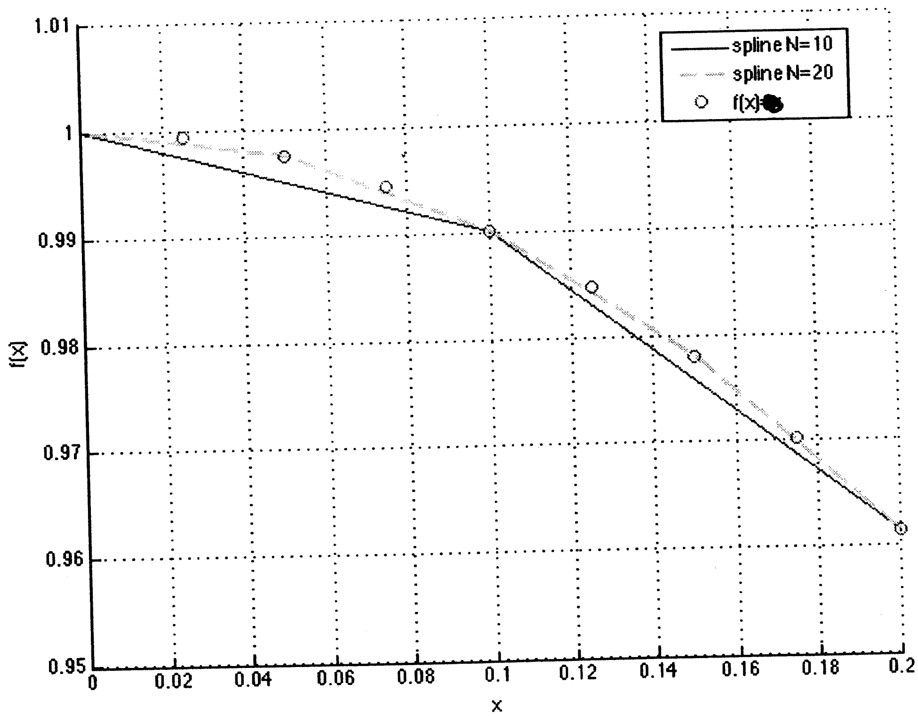
#63

```
%18.085 PS5 problem 3.2.17

clc
clear
close all

x = 0:0.025:1;
n=length(x);
fx = 1*ones(1,n)./(1*ones(1,n)+x.^2);

N=10;
xx1 = linspace(0,1,(N+1));
yy1 = spline(x,fx,xx1);

N=20;
xx2 = linspace(0,1,(N+1));
yy2 = spline(x,fx,xx2);

figure
hold on
plot(xx1,yy1,'-b')
plot(xx2,yy2,'--g','linewidth',2)
plot(x,fx,'ro')
hold off
xlabel('x')
ylabel('f(x)')
legend('spline N=10','spline N=20','f(x)=x',0)
grid on
axis on
axis([0,0.2,0.95,1.01])
```



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