18.085 Computational Science and Engineering
Homework 4

Problem 1.

2. 4. 17 has a 3x3 square grid with 9 nodes and 12 edges.

a) Count the zeros in $A^T A$

$A^T A$ matrix = the diagonal matrix (number of edges for each nodes) - the Adjacent matrix (connectivity between each nodes). Since the size of A is 12 x 9, the size of $A^T A$ is 9 x 9. Then, the number zeros in $A^T A$ is

$$N(zeros) = 9^2 - N(diagonal) - 2 \times N(edges) = 81 - 9 - 2 \times 12 = 48$$

b) Find the main diagonal

Since the diagonal matrix shows the number of connected edges for each nodes, the diagonal matrix is

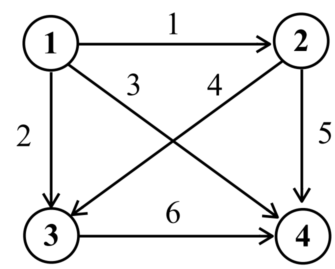
$$Diagonal \;\; Matrix = diag(2,3,2,3,4,3,2,3,2)$$

c) Find the middle row (5th row) of $A^T A$ with 3 and four -1s

Based on the definition of $A^T A$=diagonal - adjacent, the 5th row of $A^T A$ is

$$A^T A(5,\ :) = [\ 0 \quad -1 \quad 0 \quad -1 \quad 4 \quad -1 \quad 0 \quad -1 \quad 0\ ]$$

Problem 2.



The graph has 4 nodes and all 6 edges, and A is 6 by 4 matrix.

a) Find $A^T A$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

b) Ground node 4 by $u_4 = 0$. Remove column 4 so A is 6 by 3 with independent columns.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Then, $A^T A$ is linearly independent and det($A^T A$) = 16 > 0, so $A^T A$ is invertible matrix.

c) Put in random batteries b = rand(6,1). Solve $A^T A u = A^T b$ and find the average voltage at node 1, u_1 with 1000 times trial.

With the MATLAB code as like below, the average value of u_1 is -0.7463.

```
A=[-1 1 0;-1 0 1;-1 0 0;0 -1 1;0 -1 0;0 0 -1];
u = zeros(1,1000);
i = 1; lp = 1:1:1000;

for loop = lp

    b=rand(6,1);
    sol=(A'*A)\(A'*b);
    u(1,i)=sol(1,1);

    i=i+1;

end

avg = (sum(u))./1000

avg =

   -0.7463
```

Problem 3.

For the same 4 nodes graph with all 6 edges, fix $u_1 = 1$ as well as $u_4 = 0$. The vectors b and f are zero and C = I. Solve for u and w and find the total current to node 4.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ u_2 \\ u_3 \\ 0 \end{bmatrix}, \quad b = f = 0, \quad C = I$$

With the node 4 grounded, $$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ u_2 \\ u_3 \end{bmatrix}$$

Equations for voltage u and current w are

$$e = b - Au, \quad w = Ce, \quad f = A^T w \quad \Rightarrow \quad A^T A u = 0 \quad \text{and} \quad w = Au$$

$$A^T A u = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} u_2 + \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} u_3 = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3u_2 - u_3 = 1 \\ -u_2 + 3u_3 = 1 \end{cases} \quad \Rightarrow \quad u_2 = u_3 = \frac{1}{2}, \quad u = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

$$w = Au = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \\ -1 \\ 0 \\ -0.5 \\ -0.5 \end{bmatrix}$$

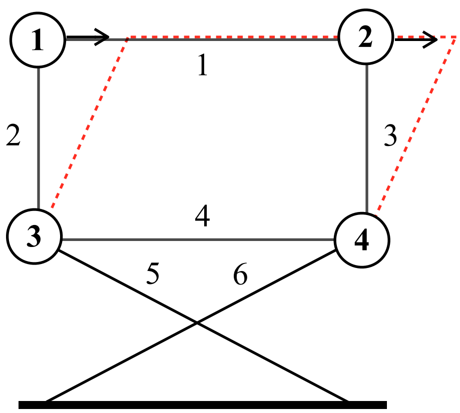
Since the node 4 is connected by edge 3, 5 and 6, the total current into node 4 is 0.5+0.5+1 = 2.

2. 7. 1

For Truss A, how many independent solutions to Au = 0 ? Draw them and also find the solution $u = \left(u_1^H, u_1^V, \ldots, u_4^H, u_4^V\right)$. What shape are A and $A^T A$? What are their first row?

Since the truss has 4 hinges with 8 displacements and 6 bars, Au = 0 has at least 8 - 6 = 2 independent solution. From the shape and configuration, there are two possible solutions for Au = 0 which make no elongation of bars.

a) Horizontal translation of hinge 1 and 2 without motion at hinge 3 and 4

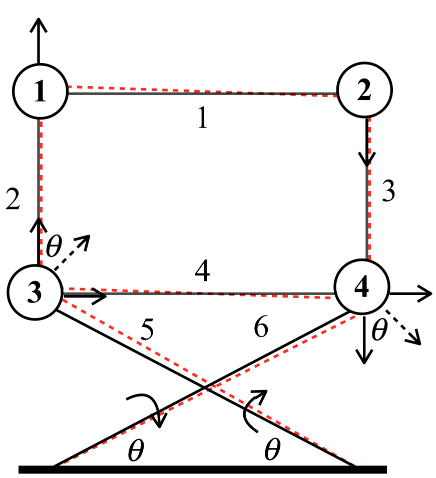


For this case, there are only horizontal motion at hinge 1 and 2, and no motion for other hinges.

$$u_1^H = u_2^H = 1 \quad \text{and} \quad u_1^V = u_2^V = u_3^H = u_3^V = u_4^H = u_4^V = 0$$

u = (1, 0, 1, 0, 0, 0, 0, 0)

b) Rotation of hinge 3 and 4 with vertical translation of hinge 1 and 2



For this case, hinge 3 and 4 rotate along bar 5 and 6 while hinge 1 and 2 vertically translate as the same amount as hinge 3 and 4 respectively.

$$u_3^H = \sin\theta,\ u_3^V = \cos\theta,\ u_4^H = \sin\theta,\ u_4^V = -\cos\theta,\ u_1^V = \cos\theta,\ u_2^V = -\cos\theta,\ u_1^H = u_2^H = 0$$

$$u = (0,\ \cos\theta,\ 0,\ -\cos\theta,\ \sin\theta,\ \cos\theta,\ \sin\theta,\ -\cos\theta)$$

For shape of A, the matrix A is 6 by 8 matrix, and $A^T A$ is 8 by 8 matrix. From the definition of A, elongation of bars $e = Au$.

$$e = Au = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \\ u_4^H \\ u_4^V \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1+\cos^2\theta & -\cos\theta\sin\theta & -1 & 0 \\ 0 & -1 & 0 & 0 & -\cos\theta\sin\theta & 1+\sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1+\cos^2\theta & \cos\theta\sin\theta \\ 0 & 0 & 0 & -1 & 0 & 0 & \cos\theta\sin\theta & 1+\sin^2\theta \end{bmatrix}$$

The first row of A is [-1 0 1 0 0 0 0 0], and the first row of $A^T A$ is [1 0 -1 0 0 0 0 0].

2. 7. 2

Truss B has 7 bars and 8 unknown displacements. What motion solves Au = 0 ? By adding one bar, can A become square and invertible? Write out row 2 of A. What is the third equation in $A^T w = f$ with right side f_2^H ?

a) There is at least 8 - 7 = 1 nonzero solution for Au = 0. When the entire truss rotates around the fixed hinge 5, there is no elongation of bars but still nonzero solution u. Hence, the rotation around the hinge 5 will solve Au = 0.

b) If we add one more bar, the number of bars is 8 with 8 unknown displacements, then the matrix A becomes 8 by 8 square matrix. However, with 8 bars, we still can rotate the entire truss around the fixed hinge 5, and this motion will still solve Au = 0. Therefore, even though A can become square matrix by adding one more bar, it is still not invertible because of the existence of nonzero solution to Au = 0.

c) The row 2 of A is following. Assuming the bar 2 is at 45 degree angle from the bar 5.

$$e_2 = \begin{bmatrix} 0 & 0 & -\sin 45^o & \cos 45^o & \sin 45^o & -\cos 45^o & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \\ u_4^H \\ u_4^V \end{bmatrix}$$

$$A(2, :) = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

d) Assuming the bar 2 at 45 degree angle (from the bar 5) and bar 4 at θ degree angle (from the ground), the third equation in $A^T w = f$ with f_2^H at right side, which is horizontal force balance at hinge 2, is

$$w_1 - w_2 \sin 45^o + w_4 \cos\theta = f_2^H ,$$

$$\text{if } \theta = \frac{1}{4}\pi, \quad w_1 - \frac{w_2}{\sqrt{2}} + \frac{w_4}{\sqrt{2}} = f_2^H$$

2. 7. 3

Truss C is a square with no support. Find 8 - 4 = 4 independent solutions to Au = 0. Find 4 sets of f's so that $A^T w = f$ has a solution. Check that $u^T f = 0$ for these four u's and f's. The force f must not activate the instabilities u.

a) Since there is no support, the rigid motions are possible in addition to mechanism.

i) Rigid horizontal translation
The entire truss horizontally translates to the right.

$$u = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^T$$

ii) Rigid vertical translation
The entire truss vertically translate to the upward.

$$u = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T$$

iii) Clockwise rotation about the hinge 3
The entire truss rotates about the hinge 3 in clockwise direction without configuration change between bars.

$$\begin{aligned} u &= \begin{bmatrix} \sin 45^o & 0 & \sin 45^o & -\cos 45^o & 0 & 0 & 0 & -\cos 45^o \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}^T \end{aligned}$$

iv) Horizontal translation of hinge 1 and 2 without motion at hinge 3 and 4
The same case as 2. 7. 1 a). Only hinge 1 and 2 translate horizontally to the right while the other two hinges remain static.

$$u = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

b) If $A^T w = f$ has nonzero solutions, external forces uniquely determine the internal bar forces and vice versa. The simplest cases are (tensile or compressive) external forces acting at both ends of each four bars. For instance, there are four sets of tensile external forces acting at each four bars.

$$\begin{cases} (a)\ f = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T : Tensile\ \ forces\ \ at\ \ bar\ \ 1 \\ (b)\ f = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}^T : Tensile\ \ forces\ \ at\ \ bar\ \ 2 \\ (c)\ f = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}^T : Tensile\ \ forces\ \ at\ \ bar\ \ 3 \\ (d)\ f = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}^T : Tensile\ \ forces\ \ at\ \ bar\ \ 4 \end{cases}$$

c) Put numbers for four u's and four f's as u_1, u_2, u_3, u_4 and f_1, f_2, f_3, f_4, we can check $u^T f = 0$ for the above four u's and f's.

$$\begin{cases} u_1^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \\ u_2^T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \\ u_3^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ u_4^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{cases} \quad \begin{cases} f_1 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ f_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}^T \\ f_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}^T \\ f_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}^T \end{cases}$$

$$\begin{aligned} u_1^T f_1 = u_1^T f_2 = u_1^T f_3 = u_1^T f_4 = 0 \\ u_2^T f_1 = u_2^T f_2 = u_2^T f_3 = u_2^T f_4 = 0 \\ u_3^T f_1 = u_3^T f_2 = u_3^T f_3 = u_3^T f_4 = 0 \\ u_4^T f_1 = u_4^T f_2 = u_4^T f_3 = u_4^T f_4 = 0 \end{aligned}$$

2. 7. 4

Truss D has how many rows and columns in the matrix A? Find column 1, with 8 elongations from a small displacement u_1^H. Draw a nonzero solution to Au = 0. Why will $A^T w = 0$ have a nonzero solution?

a) Since the truss D has 8 bars and 10 - 2 = 8 unknown displacements, the matrix A has the size of 8 by 8.

b) If there is small displacement u_1^H, the bar 1 is stretched horizontally and bar 3 is compressed horizontally while other bars are not affected by the displacement. Assume that all inclined bars at the angle of 45 degrees.

$$A(:,\ 1) = \begin{bmatrix} \cos 45^o & 0 & -\sin 45^o & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

c) We can rotate the entire truss around the fixed hinge 5 without elongations of any bars. Therefore, there is nonzero solutions for Au = 0.

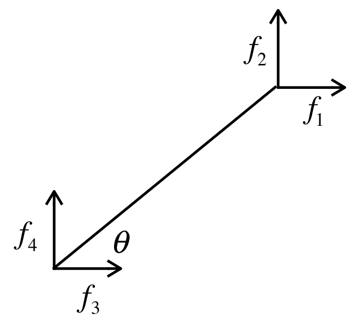
d) From the result of c), the matrix A is singular because it has nonzero solution for Au = 0. Based on the definition, $A^T w = 0$ also has nonzero solution if A is singular matrix.

$$A^T w = A^T C A u = 0 \quad \Rightarrow \quad if \ \ Au = 0 \ \ for \ \ u \neq 0, \ \ A^T C A u = 0 \ \ for \ \ u \neq 0$$

2. 7. 9

Suppose a truss consists of one bar at an angle θ with the horizontal. Sketch force f_1 and f_2 at the upper end, acting in the positive x and y directions, and corresponding forces f_3 and f_4 at the lower end. Write down the 1 by 4 matrix A_0, the 4 by 1 matrix A_0^T, and the 4 by 4 matrix $A_0^T C A_0$. This is the element matrix. For which forces can the equation $A_0^T y = f$ be solved?

a)



$$A_0 = \begin{bmatrix} \cos\theta & \sin\theta & -\cos\theta & -\sin\theta \end{bmatrix}, \quad A_0^T = \begin{bmatrix} \cos\theta & \sin\theta & -\cos\theta & -\sin\theta \end{bmatrix}^T$$

$$A_0^T C A = \begin{bmatrix} c\cos^2\theta & c\cos\theta\sin\theta & -c\cos^2\theta & -c\cos\theta\sin\theta \\ c\cos\theta\sin\theta & c\sin^2\theta & -c\sin\theta\sin\theta & -c\sin^2\theta \\ -c\cos^2\theta & -c\cos\theta\sin\theta & c\cos^2\theta & c\cos\theta\sin\theta \\ -c\cos\theta\sin\theta & -c\sin^2\theta & c\cos\theta\sin\theta & c\sin^2\theta \end{bmatrix}$$

b) To $A_0^T y = f$ be solved, external forces f's should uniquely determine internal bar forces w's and vice versa (without inducing instabilities in u). Therefore, the external forces should be matching tensile forces or compressive forces acting opposing ends of the bar. The external forces f's should have form of

$$f = k\begin{bmatrix} \cos\theta & \sin\theta & -\cos\theta & -\sin\theta \end{bmatrix}^T \text{ for an arbitrary constant k}$$