18.085 Computational Science and Engineering
Homework 3

1. 6. 27

1) Positive definiteness of M and N

Assume that H is m x m and K is n x n square, and both of them are positive definite.

$$M = \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix} \text{ and } N = \begin{pmatrix} K & K \\ K & K \end{pmatrix}$$

Positive definiteness can be shown by observing pivots for each matrices. Assume that the upper triangular matrices by row operation for H and K are U_H and U_K respective, we can write the upper triangular matrices for M and N as following.

$$U_M = \begin{pmatrix} U_H & 0 \\ 0 & U_K \end{pmatrix} \text{ and } U_N = \begin{pmatrix} U_K & U_K \\ 0 & 0 \end{pmatrix}$$

Hence, the pivots of the matrix M consists of pivots of the matrix H and K. Since H and K are positive definite and have m and n positive pivots respectively, all pivots of the matrix M is also all positive. Therefore, the matrix M is positive definite.

However, according to the upper triangular matrix of N, the matrix N has n zero pivots in addition to n pivots of the matrix K. Therefore, the matrix N is positive semidefinite rather than positive definite.

2) Connecting pivots of H and K to pivots of M and N

As shown above, pivots of the matrix M consists of pivots of H and K while pivots of the matrix N is composed of pivots of K and n zero pivots.

$$pivot(M) = pivot(H) \cup pivots(K) \text{ and } pivot(N) = pivot(K) \cup \{\underbrace{0, \cdots ,0}_{n}\}$$

3) Connection eigenvalues of H and K to eigenvalues of M and N

Assume that eigenvalues of H and K are defined as following.

$$\begin{cases} Hu_i^H = \lambda_i^H u_i^H \quad where \;\; u_i^H \;\; are \;\; eigenvectors \;\; for \;\; H, \; and \;\; \lambda_i^H \;\; are \;\; eigenvalues \;\; for \;\; H \,(for \;\; i=1 \;\ldots\; m) \\ Ku_j^K = \lambda_j^K u_j^K \quad where \;\; u_j^K \;\; are \;\; eigenvectors \;\; for \;\; H, \; and \;\; \lambda_j^K \;\; are \;\; eigenvalues \;\; for \;\; K \;(for \;\; j=1 \;\ldots\; n) \end{cases}$$

For the matrix M, we can show the eigenvalues for the matrix M based on following observations.

$$\begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} u_i^H \\ 0 \end{pmatrix} = \begin{pmatrix} Hu_i^H \\ 0 \end{pmatrix} = \lambda_i^H \begin{pmatrix} u_i^H \\ 0 \end{pmatrix} \qquad \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} 0 \\ u_j^K \end{pmatrix} = \begin{pmatrix} 0 \\ Ku_j^K \end{pmatrix} = \lambda_j^K \begin{pmatrix} 0 \\ u_j^K \end{pmatrix}$$

Therefore, the matrix M has m+n eigenvalues and corresponding eigenvectors which are composed of λ_i^H and λ_j^K for eigenvalues, and $[\; u_i^H \quad 0 \;]^T$ and $[\; 0 \quad u_j^K \;]^T$ for eigenvectors respectively. Note that all eigenvalues for H and K are positive, the matrix M also has all positive eigenvalues which guarantees the positive definiteness of the matrix M again.

$$eig(M) = \lambda_i^H \;\; and \;\; \lambda_j^K \;\; for \;\; i=1 \;\cdots\; m \;\; and \;\; j=1 \;\cdots\; n$$

For the matrix N, we can do the similar approach.

$$\begin{pmatrix} K & K \\ K & K \end{pmatrix}\begin{pmatrix} u_j^K \\ u_j^K \end{pmatrix} = \begin{pmatrix} 2Ku_j^K \\ 2Ku_j^K \end{pmatrix} = 2\lambda_j^K \begin{pmatrix} u_j^K \\ u_j^K \end{pmatrix}$$

$$\begin{pmatrix} K & K \\ K & K \end{pmatrix}\begin{pmatrix} u_k \\ -u_k \end{pmatrix} = 0 = \lambda \begin{pmatrix} u_k \\ -u_k \end{pmatrix} \text{ where } u_k \text{ are } n \times 1 \text{ matrices}$$

From the above observation, the two times of the eigenvalues of the matrix K are also n eigenvalues of the matrix N and $[\; u_j^K \quad u_j^K \;]^T$ are corresponding eigenvectors. For the second part, we can see that all $n \times 1$ u_k matrices can be decomposed into n linearly independent column matrices with size of $n \times 1$.

$$u_k = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} c_2 + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} c_{n-1} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} c_n$$

Hence, the matrix N has n zeros eigenvalues. This observation is also agrees with the previous observation that the matrix N is semi-definite.

$$eig(N) = 2\lambda_j^K \;\; and \;\; n \;\; zero \;\; eigenvaleus \;\; for \;\; j = 1 \;\cdots\; n$$

4) Cholesky factorization of M

Assume A and B are Cholesky factorization of H and K respectively.

$$chol(H) = A \text{ and } chol(K) = B \text{ where } A^T A = H \text{ and } B^T B = K$$

Then, assume that $chol(M) = C$ where the matrix C is defined as following.

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} chol(H) & 0 \\ 0 & chol(K) \end{pmatrix}$$

$$C^T C = \begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A^T A & 0 \\ 0 & B^T B \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix} = M$$

Therefore, C is right Cholesky factorization of M based on chol(H) and chol(K). Note that the matrix N does not have Cholesky factorization because it is singular.

2. 1. 3

According to the textbook, $A^T CA$ for fixed-free system is

$$A^T CA = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}$$

$\left(A^T CA\right)^{-1}$ and $A^{-1}C^{-1}(A^T)^{-1}$ can be found by row operation.

1) $\left(A^{T}CA\right)^{-1}$

$$\left(\begin{array}{ccc|ccc} c_1+c_2 & -c_2 & 0 & 1 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 & 1 & 0 \\ 0 & -c_3 & c_3 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} c_1 & 0 & 0 & 1 & 1 & 1 \\ -c_2 & c_2 & 0 & 0 & 1 & 1 \\ 0 & -c_3 & c_3 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c_1} & \frac{1}{c_1} & \frac{1}{c_1} \\ 0 & 1 & 0 & \frac{1}{c_1} & \frac{1}{c_1}+\frac{1}{c_2} & \frac{1}{c_1}+\frac{1}{c_2} \\ 0 & 0 & 1 & \frac{1}{c_1} & \frac{1}{c_1}+\frac{1}{c_2} & \frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_3} \end{array}\right)$$

$$(A^{T}CA)^{-1} = \begin{pmatrix} \frac{1}{c_1} & \frac{1}{c_1} & \frac{1}{c_1} \\ \frac{1}{c_1} & \frac{1}{c_1}+\frac{1}{c_2} & \frac{1}{c_1}+\frac{1}{c_2} \\ \frac{1}{c_1} & \frac{1}{c_1}+\frac{1}{c_2} & \frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_3} \end{pmatrix}$$

2) $A^{-1}C^{-1}(A^{T})^{-1}$

First, we need to find A^{-1} and $(A^{T})^{-1}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) = A^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = (A^{T})^{-1}$$

$$A^{-1}C^{-1}(A^{T})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c_1} & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 \\ 0 & 0 & \frac{1}{c_3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{c_1} & 0 & 0 \\ \frac{1}{c_1} & \frac{1}{c_2} & 0 \\ \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{c_1} & \frac{1}{c_1} & \frac{1}{c_1} \\ \frac{1}{c_1} & \frac{1}{c_1}+\frac{1}{c_2} & \frac{1}{c_1}+\frac{1}{c_2} \\ \frac{1}{c_1} & \frac{1}{c_1}+\frac{1}{c_2} & \frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_3} \end{pmatrix}$$

The result is the same as for $\left(A^{T}CA\right)^{-1}$.

As for special case, when $c_i = 1$ and $C = C^{-1} = I$, we can put c = 1 for above results.

$$A^T CA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \text{ and } (A^T CA)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

2. 1. 7

For fixed-fixed system with $c_1 = c_3 = c_4 = 1$ and $c_2 = 0$, the element matrix K is

$$K = A^T CA = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

To show the invertibility of the matrix K, we can find the upper triangular matrix U_K by row operation.

$$U_K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ with pivots of three 1s.}$$

Then the determinant of the matrix K is $\det(K) = 1 \times 1 \times 1 = 1 > 0$. Since the determinant is positive, the matrix K is invertible. for $Ku = f = [\ 1 \quad 1 \quad 1\ ]^T$, the solution u is

$$Ku = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} u_1 = 1 \\ u_2 - u_3 = 1 \\ -u_2 + 2u_3 = 1 \end{cases} \quad \Rightarrow \quad u_1 = 1,\ u_2 = 3,\ u_3 = 2 \ \Rightarrow \quad u = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Physically, the fixed-free system becomes two separate systems, one is a fixed-free system with one mass and one spring, and another is free-fixed system with two masses and two springs (though the second free-fixed system is in reverse configuration).

2. 1. 8

For free-free spring system, the element matrix K can be calculate by the definition $K = A^T CA$.

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix},\ C = c \text{ and } K = A^T CA = c\begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix} = c\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

the force-displacement equation for the system is

$$\begin{pmatrix} f_{1,1} \\ f_{2,1} \end{pmatrix} = c\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ where } f_{i,j} = \text{force for ith mass exerted by jth spring component}$$

(a) Assemble K for spring 2 and 3 into eq. (11) for free-free solution

In the system, there are three masses (mass 1, 2 and 3) and two springs (spring 2 and 3) with three corresponding displacement u_1, u_2 and u_3. For the first two masses connected by spring 2,

$$\begin{pmatrix} f_{1,2} \\ f_{2,2} \end{pmatrix} = c_2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \xrightarrow{expand\ \ for\ \ 3\ \ masses} \begin{pmatrix} f_{1,2} \\ f_{2,2} \\ 0 \end{pmatrix} = \begin{pmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

For the masses 2 and 3 connected by spring 3,

$$\begin{pmatrix} f_{2,3} \\ f_{3,3} \end{pmatrix} = c_3\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \xrightarrow{expand\ \ for\ \ 3\ \ masses} \begin{pmatrix} 0 \\ f_{2,3} \\ f_{3,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Combining two results, the force-displacement for the free-free system is

$$\begin{bmatrix} f_{1,2} \\ f_{2,2} + f_{2,3} \\ f_{3,3} \end{bmatrix} = \begin{pmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

which is the same as the eq. (11).

(b) Include K for spring 1 to find K for fixed-free system as the eq. (8).

For this system, there is additional spring 1 which is fixed at one end and connected to the mass 1 at the other end. For mass 1 and spring 1,

$$\begin{pmatrix} 0 \\ f_{1,1} \end{pmatrix} = c_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ u_1 \end{pmatrix} \xrightarrow{\exp and} \begin{pmatrix} 0 \\ f_{1,1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

If we simplify the matrix by eliminating zero row and column,

$$\begin{pmatrix} f_{1,1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Combining this result with the result in (a), we can find the force-displacement relationship in the fixed-free system.

$$\begin{pmatrix} f_{1,1} + f_{1,2} \\ f_{2,2} + f_{2,3} \\ f_{3,3} \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

which is the same as the eq. (8).

(c) Place spring 4 to find K for fixed-fixed solution as the eq. (7).

For this system, there is additional spring 4 which is connected to the mass 3 at one end and fixed at the other end. For mass 3 and spring 4,

$$\begin{pmatrix} f_{3,4} \\ 0 \end{pmatrix} = c_4 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_3 \\ 0 \end{pmatrix} \xrightarrow{\exp and} \begin{pmatrix} 0 \\ 0 \\ f_{3,4} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{pmatrix}$$

If we simplify the matrix by eliminating zero row and column,

$$\begin{pmatrix} 0 \\ 0 \\ f_{3,4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Combining this result with the result in (b), we can find the force-displacement relationship in the fixed-fixed system.

$$\begin{pmatrix} f_{1,1} + f_{1,2} \\ f_{2,2} + f_{2,3} \\ f_{3,3} + f_{3,4} \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

which is the same as the eq. (7).

2. 3. 7

With b = (4, 1, 0, 1) at x = (0, 1, 2, 3), solve normal equation for coefficient $\hat{u} = (C,\ D)$ in the nearest line C + Dx.

The system equation is

$$Au = b \qquad \Rightarrow \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

For the least square solution $\hat{u}$, $A^T A\hat{u} = A^T b$ is satisfied.

$$A^T A\hat{u} = A^T b \ \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \ \text{where} \ (A^T A)^{-1} = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}^{-1} = \frac{1}{20} \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}$$

$$\hat{u} = \begin{pmatrix} C \\ D \end{pmatrix} = (A^T A)^{-1} A^T b = \frac{1}{20} \begin{pmatrix} 14 & -6 \\ -6 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Therefore, C = 3 and D = -1, and the nearest line is $y = 3 - x$.

2. 3. 8

In the previous problem, the projection $P = A\hat{u}$. Check that those four numbers do lie on the line C+Dx, and compute the error $e = b - P = b - A\hat{u}$ and verify $A^T e = 0$.

From the result of 2. 3. 7,

$$P = A\hat{u} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

It is obvious that points (0, 3), (1, 2), (2, 1) and (3, 0) lie on the line b = 3 - x. For the error term,

$$e = b - P = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \text{ where } A^T e = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. 3. 24

Find the plane that gives the best fit to the 4 values b = (0 ,1 ,3, 4) at the corners (1, 0), (0, 1), (-1, 0) and (0, -1) of a square. For the equation C + Dx + Ey = b, $Au = b$ where $u = (C, D, E)$. At the center of the square (0, 0), show that C + Dx + Ey = avg(b) = avg(0, 1, 3, 4).

The system equation is

$$Au = b \qquad \Rightarrow \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$$

For the least square solution $\hat{u}$, $A^T A\hat{u} = A^T b$ is satisfied.

$$A^T A\hat{u} = A^T b \;\;\Rightarrow\;\; \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix} \text{ where } (A^T A)^{-1} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\hat{u} = \begin{pmatrix} C \\ D \\ E \end{pmatrix} = (A^T A)^{-1} A^T b = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$$

Therefore, C = 2, D = $-\frac{3}{2}$ and E = $-\frac{3}{2}$, and the best fit plane is $2-\frac{3}{2}x-\frac{3}{2}y=b$. To check the average condition,

$$b(0,0) = 2-\frac{3}{2}\cdot 0-\frac{3}{2}\cdot 0 = 2 = \frac{0+1+3+4}{4} = Avg(b)$$

2. 4. 1

What are the incidence matrices $A_{triangle}$ and A_{square} for the graphs? Find $A^T A$.

1) For triangular graph

$$A_{triangle} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \text{ where } A^T A = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

2) For square graph

$$A_{square} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \text{ where } A^T A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

2. 4. 2

Find all vectors in the nullspace of $A_{triangle}$ and its traspose.

From the result of 2. 4. 1, for $A_{triangle}$,

$$Au = 0 \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} u_1 + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u_3$$

$$-u_1 + u_2 = 0,\ -u_2 + u_3 = 0 \text{ and } -u_1 + u_3 = 0 \text{ that is } u_1 = u_2 = u_3 \Rightarrow N(u) = \begin{pmatrix} c \\ c \\ c \end{pmatrix}$$

Hence the nullspace solution is $N(u) = [\ c \quad c \quad c\ ]^T$ for arbitrary constant c.

For its transpose $(A_{triangle})^T$,

$$A^T u = 0 \Rightarrow \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} u_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} u_3$$

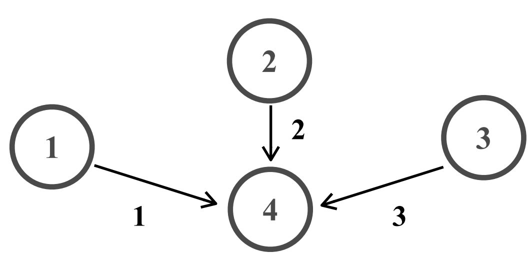
$$u_1 + u_3 = 0,\ -u_2 + u_3 = 0 \text{ and } u_2 + u_3 = 0 \text{ that is } u_1 = u_2 = -u_3 \Rightarrow N(u) = \begin{pmatrix} c \\ c \\ -c \end{pmatrix}$$

Hence the nullspace solution is $N(u) = [\ c \quad c \quad -c\ ]^T$ for arbitrary constant c.

2. 4. 7

What is $K = A^T CA$ for the four-node tree with all three edges into node 4? Ground a node to find the reduced (invertible) K and det(K).

The four-note tree for this system is following.



$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

For the weight matrix C = diag(c_1 , c_2 , c_3)

$$K = A^T C A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c_1 + c_2 + c_3 \end{pmatrix}$$

To ground node 4, we have to eliminate the fourth row and column from the matrix K.

$$K_{reduced} = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \quad \text{where} \quad \det(K_{reduced}) = c_1 \times c_2 \times c_3 = c_1 c_2 c_3$$

Note that, det(K) = 0 because the matrix K is linearly dependent (with rank(K) = 4 - 1= 3 < 4) and, therefore, singular.