Problem Set 2

Problem 1, 1.4.7

By linearity of the equation $-u'' = f$, we get a general solution for $f(x) = \delta(x - \frac{1}{3}) - \delta(x - \frac{2}{3})$ by adding up the general solutions $-R(x - \frac{1}{3})$, $R(x - \frac{2}{3})$ (ramps) for right-hand sides $\delta(x - \frac{1}{3})$, $-\delta(x - \frac{2}{3})$ and a general solution of $u'' = 0$, $x \mapsto Ax + B$, for $A, B \in \mathbb{R}$. Fitting this to the boundary conditions yields $u'(0) = A = 0$ and $u'(1) = A = 0$, so $A = 0$ is sufficient to satisfy both of them. Hence,

$$u(x) = R(x - \frac{1}{3}) - R(x - \frac{2}{3}) + B = \begin{cases} B, & 0 \leq x \leq \frac{1}{3}, \\ B - (x - \frac{1}{3}), & \frac{1}{3} < x \leq \frac{2}{3}, \\ B - \frac{1}{3}, & \frac{2}{3} < x \leq 1, \end{cases} \quad B \in \mathbb{R}. \tag{1}$$

Problem 2, 1.4.9

$$u(x) = \int_0^x (1 - x) a \,\mathrm{d}a + \int_x^1 (1 - a) x \,\mathrm{d}a = \frac{1}{2}(1 - x) a^2 \Big|_0^x - \frac{1}{2}(1 - a)^2 x \Big|_x^1 \tag{2}$$

$$= \frac{1}{2}\left(x^2 - x^3 + (1 - x)^2 x\right) = \frac{1}{2}\left(x - x^2\right), \tag{3}$$

which is the solution to $-u'' = 1$ for fixed-fixed boundary conditions from class.

Problem 3, 1.4.11

Integrating with 0 as end-point yields

$$Q(x) = \int_0^x R(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{2}x^2, & x > 0, \end{cases} \qquad C(x) = \int_0^x Q(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{6}x^3, & x > 0. \end{cases} \tag{4}$$

Since these functions are iterated integrals, they are derivatives of each other. By inspection, $C'(x) = Q(x)$, $C''(x) = R(x)$, $C'''(x) = S(x)$, so the first two derivatives are continuous, while the third one isn't.

We can plot C and Q with

```
x = linspace(-2, 2, 100);
plot(x, (x > 0).*0.5.*x.^2, x, (x > 0).*x.^3/6);
axis([-2 2 -0.5 2]);
legend('Q(x)', 'C(x)');
```

Problem 4, 1.5.9

$$\Delta_-^T \Delta_- = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = K_3, \tag{5}$$

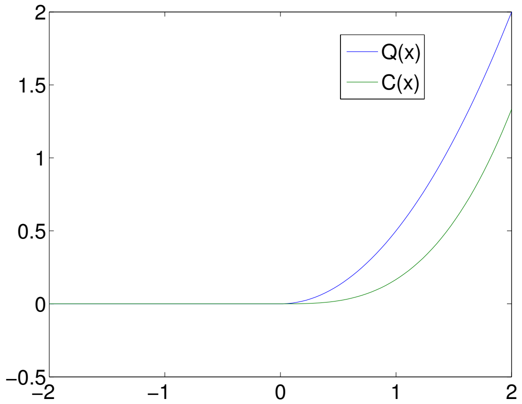

Figure 1: Plot of $Q(x)$, $C(x)$

$$\Delta_{-}\Delta_{-}^{T} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} = B_4 \tag{6}$$

And we can check using

```
e = ones(3,1);
DeltaMin = spdiags([e -e], [0, -1], 4, 3);
lambda1 = eig(DeltaMin*DeltaMin');
lambda2 = eig(DeltaMin'*DeltaMin);
```

that

$$\lambda_1 = \begin{pmatrix} 0.0000 \\ 0.5858 \\ 2.0000 \\ 3.4142 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0.5858 \\ 2.0000 \\ 3.4142 \end{pmatrix}. \tag{7}$$

Problem 5, 1.6.3

For any matrix A, $A^T A$ will at least be positive semi-definite by

$$u^T A^T A u = \|Au\|^2 \geq 0, \quad u \in V. \tag{8}$$

On the other hand, Gaussian elimination of A yields

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

from where we see that $\ker A = \operatorname{span}\{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T\}$. For elements u in that span, we'll also have $Cu = A^T A u = 0$, and by (8), these are all the elements in $\ker C$. Therefore, C is only positive semi-definite.

Problem 6, 1.6.9

Let's call the first A A_1 and the second one A_2. We have that the first upper left determinant of the first A_1 is $1 > 0$, and $\det(A_1) = 9 - b^2 > 0 \Leftrightarrow |b| < 3$, so it is positive definite for these b. For the A_2, the upper left determinant is $2 > 0$, and $\det(A_2) = 2c - 16 > 0 \Leftrightarrow c > -8$, so for these c, it is positive definite.

Row reduction on A_1 yields

$$\begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \xrightarrow{\text{row } 2 - b \times \text{row } 1} \begin{bmatrix} 1 & b \\ 0 & 9 - b^2 \end{bmatrix}, \tag{10}$$

which means that the LDL^T decomposition is

$$A_1 = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}. \tag{11}$$

Similarly, row reducing A_2 leads to

$$\begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \xrightarrow{\text{row } 2 - 2 \times \text{row } 1} \begin{bmatrix} 2 & 4 \\ 0 & c - 8 \end{bmatrix}, \tag{12}$$

so

$$A_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

Problem 7, 1.6.20

First, note that with

$$O = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad \text{we have } A = O\Lambda O^T \text{ and } O^T O = I. \tag{14}$$

By the multiplicativity of the determinant, $\det(A) = \det(O)\det(\Lambda)\det(O^T)$, and since $1 = \det(O^T O)$, $\det(O) = \det(O^T)^{-1}$, so $\det(A) = \det(\Lambda) = 10$. Moreover, the eigenvectors of A are the column vectors of O, since multiplying either one of them with O^T first yields a vector of the standard basis (by orthogonality of O), which is an eigenvector of Λ, that subsequently gets mapped back to the column of O. (In other words, $AO = O\Lambda$, which is an eigenvalue decomposition.) That means the eigenvector, eigenvalue pairs are

$$\lambda_1 = 2, \quad v_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \lambda_2 = 5, \quad v_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}. \tag{15}$$

A is positive definite because all eigenvalues are > 0.

Problem 8, Master Equations

a) Denoting $D = \operatorname{diag}(d_1, \ldots, d_5)$, the product becomes

$$DAD^{-1} = \begin{bmatrix} -16 & d_1 d_2^{-1} & 0 & 0 & 0 \\ 16 d_2 d_1^{-1} & -10 & 2 d_2 d_3^{-1} & & \\ 0 & 9 d_3 d_2^{-1} & -6 & 3 d_3 d_4^{-1} & 0 \\ 0 & 0 & 4 d_4 d_3^{-1} & -4 & 4 d_4 d_5^{-1} \\ 0 & 0 & 0 & d_5 d_4^{-1} & -4 \end{bmatrix}. \tag{16}$$

Setting the off-diagonals equal yields four equations,

$$d_1^2 = 16 d_2^2, \quad 2 d_2^2 = 9 d_3^2, \quad 3 d_3^2 = 4 d_4^2, \quad 4 d_4^2 = d_5^2. \tag{17}$$

Picking an arbitrary $d_1 \neq 0$, for example $d_1 = 4$, we can solve iteratively to get (as one possible choice out of many):

$$d_1 = 4, \quad d_2 = 1, \quad d_3 = \frac{\sqrt{2}}{3}, \quad d_4 = \frac{1}{\sqrt{6}}, \quad d_5 = \sqrt{\frac{2}{3}}, \tag{18}$$

which renders DAD^{-1} symmetric. An eigenvalue decomposition for this matrix, $DAD^{-1} = O\Lambda O^T$ with O orthogonal, in turn yields one for A, $A = D^{-1} O \Lambda (D^{-1} O)^{-1}$, so all eigenvalues of A are real.

b) Executing the code

```
Ns = [5, 50, 100];
for N = Ns
    b = 0:N−1;
    f = b.^2;
    f = fliplr(f);
    s = b+f;
    A = spdiags([f' −s' b'], [−1 0 1], N, N);
    e = eig(full(A));
    plot(e, '.');
    title(sprintf('N = %i', N))
    saveas(gcf, sprintf('p08−%i.eps', N), 'epsc');
end
```

yields

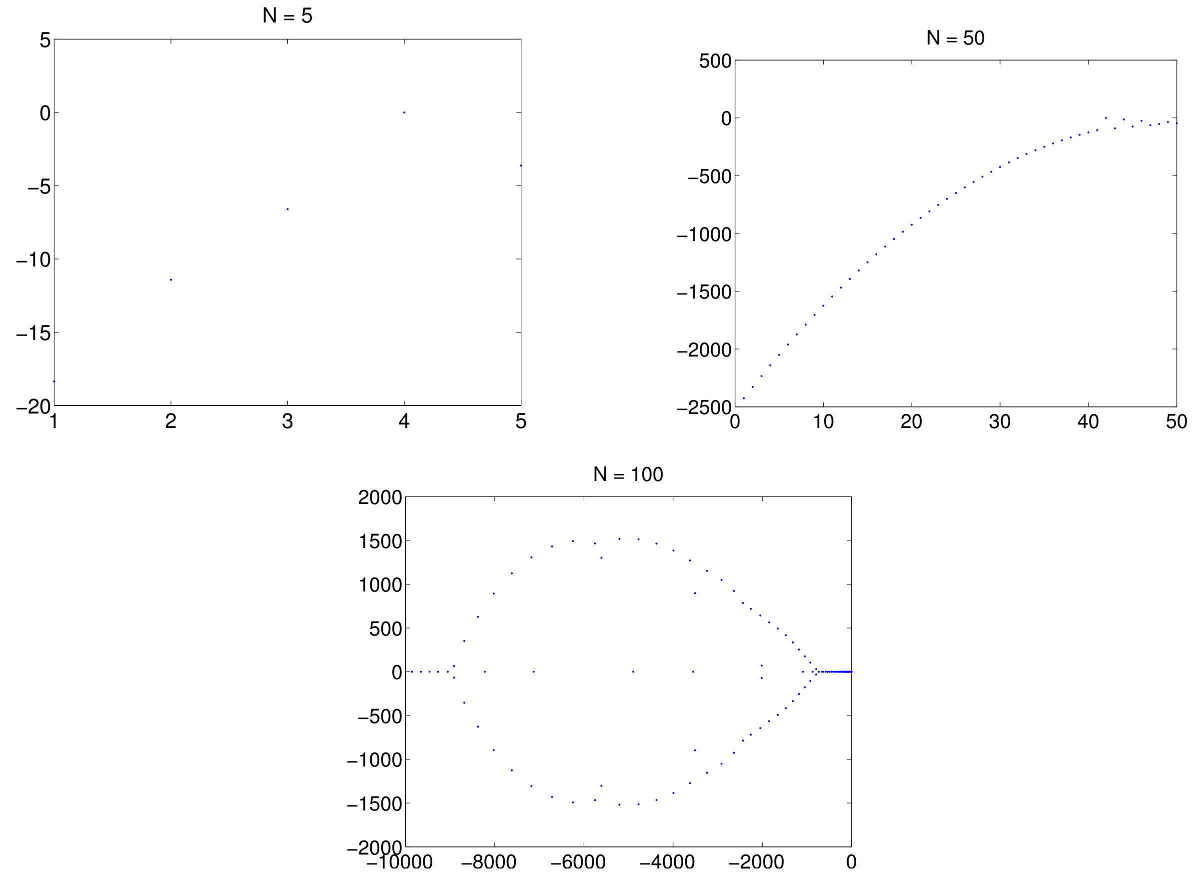

Figure 2: Eigenvalue plots