Problem Set 1

Problem 1

a)

$$\begin{bmatrix}1\\3\\2\end{bmatrix}\begin{bmatrix}1 & 3 & 2\end{bmatrix} = \begin{bmatrix}1 & 3 & 2\\3 & 9 & 6\\2 & 6 & 4\end{bmatrix}. \tag{1}$$

As for why K is singular, one could argue that it has a non-trivial nullspace, which we see in part b).

b) $Ku = 0$ if and only if $Au = 0$, and $A = \begin{bmatrix}1 & 3 & 2\end{bmatrix}$ is already triangular, so we immediately see that

$$Ku = 0 \Leftrightarrow u \in \operatorname{span}\left(\begin{bmatrix}-3\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}\right). \tag{2}$$

Note that when you are asked to compute a nullspace, it would be nice to provide a basis—in some psets, there were 4 linearly dependent vectors given that indeed span the nullspace, but two would have been sufficient to characterize it.

c)

$$K^4 = (A'A)^4 = A'(AA')^3A = 14^3K = \begin{bmatrix}2744 & 8232 & 5488\\8232 & 24696 & 16464\\5488 & 16464 & 10976\end{bmatrix}. \tag{3}$$

Note that this simple idea—computing matrix products in a numerically opportune order—is also at the heart of the adjoint method.

Problem 2

a) For cubes:

$$(n+1)^3 - 2n^3 + (n-1)^3 = n^3 + 3n^2 + 3n + 1 - 2n^3 + n^3 - 3n^2 + 3n - 1 = 6n, \quad n \in \mathbb{N}, \tag{4}$$

which is the correct second derivative, if we interpret the input as polynomial over $\mathbb{R}$, for example.

For quartics:

$$(n+1)^4 - 2n^4 + (n-1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 - 2n^4 + n^4 - 4n^3 + 6n^2 - 4n + 1 = 12n^2 + 2, \quad n \in \mathbb{N}, \tag{5}$$

which is off by 2 from the correct derivative.

b) Squares:

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1, \tag{6}$$

$$\frac{1}{2}\left((n+1)^2 - (n-1)^2\right) = \frac{1}{2}\left(n^2 + 2n + 1 - n^2 + 2n - 1\right) = 2n, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2\bigg|_{x=n} = 2n, \quad n \in \mathbb{N}. \tag{8}$$

Cubics:

$$(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1, \tag{9}$$

$$\frac{1}{2}\left((n+1)^3 - (n-1)^3\right) = \frac{1}{2}\left(n^3 + 3n^2 + 3n + 1 - n^3 + 3n^2 - 3n + 1\right) = 3n^2 + 1, \tag{10}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x} x^3 \right|_{x=n} = 3n^2, \quad n \in \mathbb{N}. \tag{11}$$

Quartics:

$$(n+1)^4 - n^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 = 4n^3 + 6n^2 + 4n + 1, \tag{12}$$

$$\frac{1}{2}\left((n+1)^4 - (n-1)^4\right) = \frac{1}{2}\left(n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 + 4n^3 - 6n^2 + 4n - 1\right) = 4n^3 + 4n, \tag{13}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x} x^4 \right|_{x=n} = 4n^3, \quad n \in \mathbb{N}. \tag{14}$$

Problem 3

Use the ansatz from class, $u(x) = -R(x-a) + Ax + B$ and fit to the boundary conditions: $0 = u(0) = B$, $0 = u'(1) = A - 1$, so $A = 1$. Combined, we get

$$u(x) = \begin{cases} x, & x \in [0, a), \\ a, & x \in [a, 1], \end{cases} \tag{15}$$

which means u first grows linearly and at a becomes constant.

We can plot it for e.g. $a = 0.4$ with

```
a = 0.4;
x = linspace(0,1,100);
plot(x, (x < a).*x + (x >= a).*a);
```

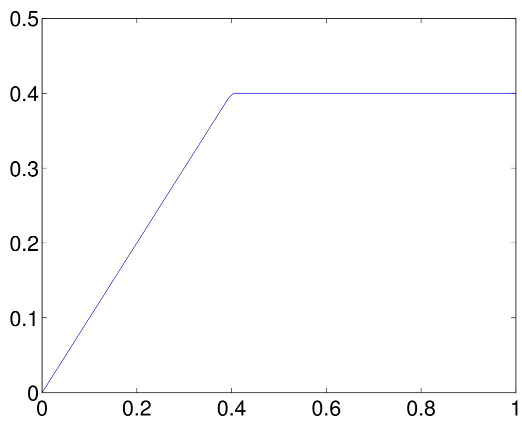

Figure 1: Plot of u for $a = 0.4$

Problem 4

For example, apply centered second differences twice to get

$$\begin{aligned}\frac{1}{(h^2)^2}&((u_{i+2}-2u_{i+1}+u_i)-2(u_{i+1}-2u_i+u_{i-1})+(u_i-2u_{i-1}+u_{i-2}))\\ &=\frac{1}{h^4}(u_{i+2}-4u_{i+1}+6u_i-4u_{i-1}+u_{i-2}),\end{aligned}\tag{16}$$

the desired coefficients. For the analysis of the corresponding matrix, we sometimes consider it without the h^{-4} factor, but when actually solving a problem, we have to keep it there.

Problem 5

We can solve it with Matlab:

```
n = 5;
h = 1/n;
e = ones(n-1, 1);

% Generate K and D (we choose centered first differences) as sparse
% matrices
K = spdiags([-e 2*e -e], -1:1, n-1, n-1);
D = spdiags([-e zeros(n-1,1) e], -1:1, n-1, n-1)/2;

% Right-hand side is just ones
b = ones(n-1,1);
u = (1/(h^2) * K + 1/h * D)\b;
u = [0; u; 0];
```

This yields

$$u=\begin{bmatrix}0\\0.0714\\0.1141\\0.1220\\0.0871\\0\end{bmatrix}.\tag{17}$$

Problem 6

By Gaussian elimination (or using Matlab), we see

$$T_3=\begin{bmatrix}3&2&1\\2&2&1\\1&1&1\end{bmatrix},\quad T_4=\begin{bmatrix}4&3&2&1\\3&3&2&1\\2&2&2&1\\1&1&1&1\end{bmatrix},\tag{18}$$

whence we guess the general form

$$(T_n^{-1})_{i,j}=\begin{cases}n-i+1, & i\geq j,\\ n-j+1, & j>i,\end{cases},\quad 1\leq i,j\leq n.\tag{19}$$

In order to verify that this is indeed the inverse, look at what happens when we multiply by T_n and compute $(T_nT_n^{-1})_{i,j}$: It is enough to look at one row of T_n, and we note that there are only three cases for the corresponding entries in T_n^{-1}, namely $(k,\, k+1,\, k+2)'$, $(k,\, k+1,\, k+1)'$ and $(k,\, k,\, k)'$ for some integer k, for the interior rows $1 < i < n$. Checking the result yields:

$$\begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} k \\ k+1 \\ k+2 \end{bmatrix} = -k + 2(k+1) - (k+2) = 0 \tag{20}$$

$$\begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} k \\ k+1 \\ k+1 \end{bmatrix} = -k + 2(k+1) - (k+1) = 1 \tag{21}$$

$$\begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} k \\ k \\ k \end{bmatrix} = -k + 2k - k = 0. \tag{22}$$

Since the case $(k,\, k+1,\, k+1)'$ corresponds to the diagonal element, we have the correct result for the identity matrix.

The calculations for the first and the last row, $i = 1$ or $i = n$, respectively, are similar.