

18.085 SUMMER 2013 - QUIZ 3 - AUGUST 15, 2013

YOUR NAME: \_\_\_\_\_

YOUR SCORE: \_\_\_\_\_ / 100 + \_\_\_\_\_ / 20 extra credit

**THE QUIZ IS OPEN BOOK, OPEN NOTES AND NO CALCULATORS**

**GRADING:**

(1) 1. \_\_\_\_\_

(2) 2. \_\_\_\_\_ + \_\_\_\_\_ **BONUS POINTS**

(3) 3. \_\_\_\_\_

(4) 4. \_\_\_\_\_

(1) (20 points.)

The dft of the signal  $x$  is

$$\hat{x} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

What signal is sampled in  $x$ ?

Sketch the real and the imaginary part of  $x$ . (With our usual notation, here we have that  $N=11$ ). **Solution:**

As

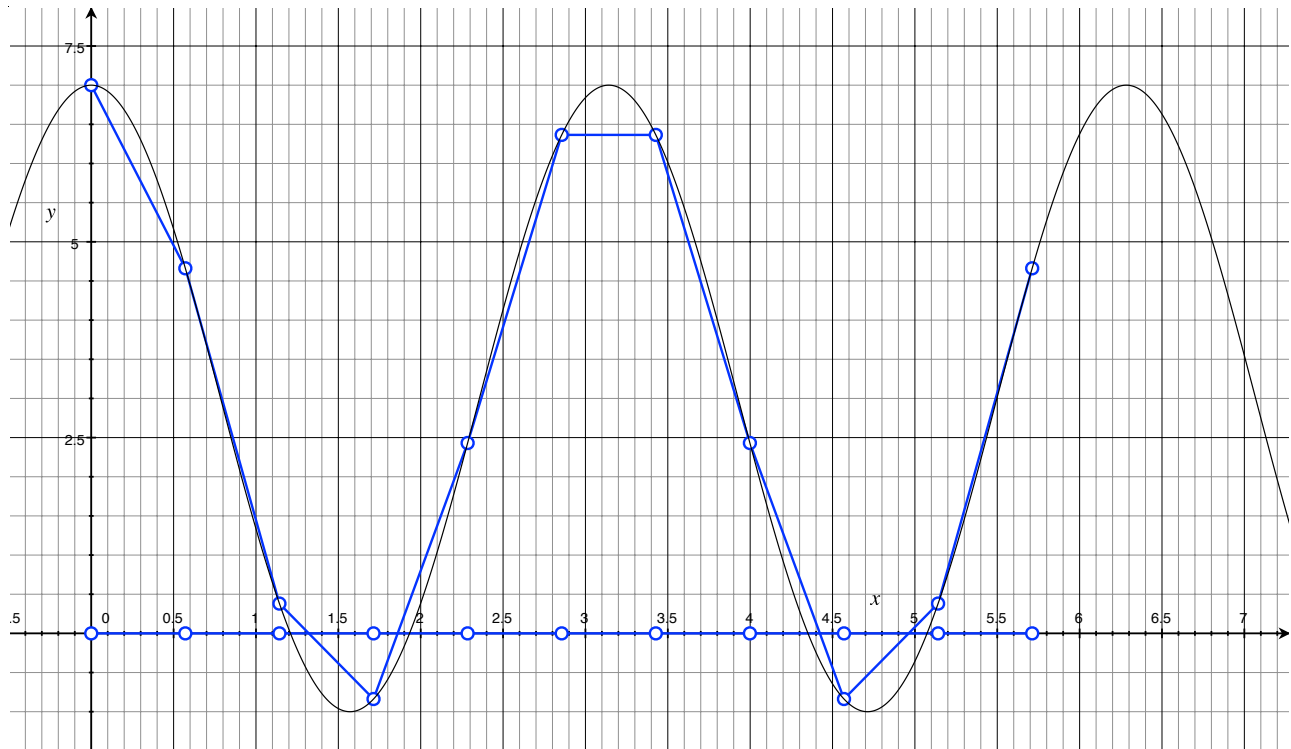
$$x = G_N \hat{x},$$

it follows that

$$x = \sum_{k=0}^{N-1} \hat{x}_k v_k = 3v_0 + 2v_2 + 2v_9 =$$

$$3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ e^{1 \cdot 2 \cdot 2\pi i / 11} \\ e^{2 \cdot 2 \cdot 2\pi i / 11} \\ \vdots \\ e^{10 \cdot 2 \cdot 2\pi i / 11} \end{pmatrix} + 2 \begin{pmatrix} 1 \\ e^{-1 \cdot 2 \cdot 2\pi i / 11} \\ e^{-2 \cdot 2 \cdot 2\pi i / 11} \\ \vdots \\ e^{-10 \cdot 2 \cdot 2\pi i / 11} \end{pmatrix} = \begin{pmatrix} 3 + 4 \\ 3 + 4 \cos\left(\frac{2 \cdot 2\pi}{11}\right) \\ 3 + 4 \cos\left(\frac{4 \cdot 2\pi}{11}\right) \\ \vdots \\ 3 + 4 \cos\left(\frac{20 \cdot 2\pi}{11}\right) \end{pmatrix}.$$

The graph of the real part is the graph of a  $f(t) = 3 + 4 \cos(2t)$  and the points we are sampling are those corresponding to  $x = 0, \frac{2\pi}{11}, \frac{2 \cdot 2\pi}{11}, \dots, \frac{10 \cdot 2\pi}{11}$ .



(2) (15 + 15 + 10 points +10 + 10 bonus points.)

Which of these equations can be solved? If you can solve them, then show a solution, otherwise explain why it is not possible to find a solution.

a)

$$\operatorname{div} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = x^2 - 3\lambda y,$$

where  $\lambda$  is a fixed real parameter.

b) The Laplace equation on the unit circle, with boundary condition,

$$-\Delta u = 0, \quad u(1, \theta) = \sin(2\theta) + \cos(6\theta) + 1.$$

c)

$$\operatorname{div} \begin{pmatrix} -\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \log x^2 - 3y,$$

**Solution:**

a) A solution to

$$\operatorname{div} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \Delta u = x^2 - 3\lambda y$$

is given by  $u(x, y) = \frac{x^4}{12} - \lambda \frac{x^2 y}{2}$ .

b) A solution to

$$-\Delta u = 0, \quad u(1, \theta) = \sin(2\theta) + \cos(6\theta) + 1.$$

is given by

$$u(r, \theta) = r^2 \sin(2\theta) + r^6 \cos(6\theta) + 1$$

as shown in the book at page .

c) The equation

$$\operatorname{div} \begin{pmatrix} -\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x} \end{pmatrix} = \log x^2 - 3y,$$

has no solution, since

$$\operatorname{div} \begin{pmatrix} -\frac{\partial s}{\partial y} \\ \frac{\partial s}{\partial x} \end{pmatrix} = -\frac{\partial^2 s}{\partial x \partial y} + \frac{\partial^2 s}{\partial y \partial x} = 0.$$

**Caveat:** the next two are bonus questions.

- d1) Find a family of curves  $s(x, y) = C$  that is everywhere orthogonal to the family of curves  $u(x, y) = e^x(\cos(y))$ .
- d2) Can you tell for which complex function the function  $u(x, y)$  in d1) is the real part?

**Solution:**

- d1) As  $\text{div}(\nabla u) = e^x \cos(y) - e^x \cos(y) = 0$ , then  $u$  admits a stream function  $s$ , which is determined by the two conditions

$$\begin{aligned}\frac{\partial s}{\partial y} &= \frac{\partial u}{\partial x} = e^x \cos(y), \\ -\frac{\partial s}{\partial x} &= \frac{\partial u}{\partial y} = -e^x \sin(y).\end{aligned}$$

Then,

$$\begin{aligned}s &= \int \frac{\partial s}{\partial y} dy + F(x) = \int e^x \cos(y) dy + F(x) = e^x \sin(y) + F(x), \\ s &= \int \frac{\partial s}{\partial x} dx + G(y) = \int e^x \sin(y) dx + G(y) = e^x \sin(y) + G(y).\end{aligned}$$

Hence  $s(x, y) = e^x \sin(y)$  is a solution.

- d2) The complex function we are looking for has  $u$  as real part and  $s$  as imaginary part, hence

$$\begin{aligned}h(x, y) &= u(x, y) + is(x, y) = e^x \cos(y) + ie^x \sin(y) \\ &= e^x(\cos(y) + i \sin(y)) = e^z\end{aligned}$$

where  $z = x + iy$ .

(3) (15 points). Let  $f(x) = |x| - g(x)$  on  $[-\pi, \pi]$ , where  $g(x)$  is defined as follows,

$$g(x) = \begin{cases} 15 & \text{for } x \in [0, \pi] \\ -1 & \text{for } x \in (-\pi, 0). \end{cases}$$

Find the Fourier series of  $f(x)$ , when  $f$  is extended periodically to the real line, by the rule  $f(2\pi + x) = f(x)$ .

**Solution:**

We can rewrite  $g$  as  $g(x) = 7 + 8SW(x)$ , where

$$SW(x) = \begin{cases} 1 & \text{for } x \in [0, \pi] \\ -1 & \text{for } x \in (-\pi, 0). \end{cases}$$

The Fourier series of  $SW(x)$ , that we saw in class multiple times, is

$$\frac{4}{\pi} \sum_{\mathbb{N} \ni k, k \text{ odd}} \frac{\sin(kx)}{k},$$

hence the Fourier series of  $g(x)$  is

$$g(x) = 7 + \frac{32}{\pi} \sum_{\mathbb{N} \ni k, k \text{ odd}} \frac{\sin(kx)}{k}.$$

$h(x) = |x|$  is an even function, hence its Fourier series will be a series of cosines,  $\sum_{k=0}^{\infty} c_k \cos(kx)$ . Let us compute it.

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} -|x| dx = \frac{\pi}{2} \\ c_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} -|x| \cos(kx) dx = \begin{cases} 0 & \text{for } k \text{ even} \\ -\frac{4}{\pi k^2} & \text{for } k \text{ odd.} \end{cases} \end{aligned}$$

Hence the Fourier series of  $f(x)$  is

$$f(x) = \frac{\pi}{2} - 7 + \sum_{\mathbb{N} \ni k, k \text{ odd}} -\frac{32 \sin(kx)}{\pi k} - \frac{4 \cos(kx)}{\pi k^2}.$$

(4) (5 + 5 + 5 + 5 + 5 points)

a) Consider the function  $f(x) = e^{-x}$ ,  $x \in [-\pi, \pi]$  which is extended periodically to the real line, by the rule  $f(2\pi + x) = f(x)$ . Draw the graph in  $[-2\pi, 2\pi]$ .

The function satisfies the following differential equation

$$(0.1) \quad \frac{d}{dx}f(x) + f(x) = g(x),$$

for some function  $g(x) : [-\pi, \pi] \rightarrow \mathbb{R}$ . Find such  $g$ .

b) Compute the complex Fourier series of  $f(x)$ ,

$$f(x) = \sum_{k \in \mathbb{Z}} d_k e^{ikx},$$

in the standard way.

**Solution:**

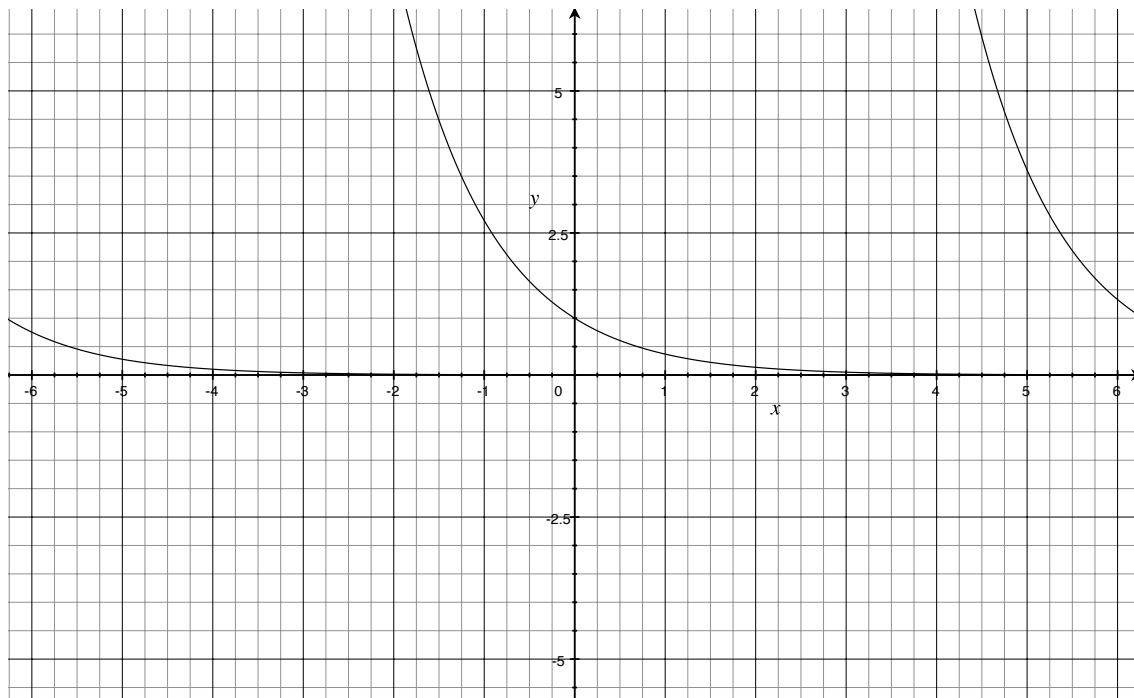


FIGURE 1. The graph of  $f$ .

a) For the differential equation, we have that

$$\frac{d}{dx}e^{-x} + e^{-x} = -e^{-x} + e^{-x} = 0,$$

hence  $g(x)$  is the constantly 0 function.

b)

$$\begin{aligned}d_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(1+ik)x} dx = -\frac{1}{2\pi} \frac{e^{-(1+ik)x}}{(1+ik)} \Big|_{-\pi}^{\pi} = \\ &= \frac{1}{2\pi} \left[ \frac{e^{(1+ik)\pi}}{(1+ik)} - \frac{e^{-(1+ik)\pi}}{(1+ik)} \right] = \frac{\sinh((1+ik)\pi)}{\pi(1+ik)}.\end{aligned}$$



- c) Try to compute the Fourier coefficients of  $f$  using the differential equation (0.1). Do you get the same result as in part b)?
- d) Explain the reason for the answer you gave in part c).
- e) Compute

$$\sum_{k \in \mathbb{Z}} |d_k|^2.$$

**Solution:**

- c) Applying the differential equations to the Fourier coefficients yields the following equations

$$d_k(ik) + d_k = 0, \text{ for all } k \in \mathbb{Z}, k \neq 0.$$

These equations imply that for  $|k| \neq 1$ , then  $d_k = 0$ , which is different from what we obtained in part b).

- d) The problem is that  $f$  is not differentiable when we look at it as a function on the real line. Hence, as the differential equations (0.1) holds only in  $(-\pi, \pi)$  and the discontinuity at  $-\pi$  is the source of the discrepancy in the results between part b) (the right one) and part c) (the wrong one).
- e)

$$\sum_{k \in \mathbb{Z}} |d_k|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2x} dx = \frac{e^{2\pi} - e^{-2\pi}}{4\pi}.$$