18.085 Exam 3 Professor Strang December 5, 2013

Your PRINTED NAME is: SOLUTIONS

Your CLASS NUMBER is: ______

1. (20 points) The periodic Hartley transform is also called the **cosine and sine** transform because its basis functions for $-\pi < x \le \pi$ are

$$\mathrm{CAS}_n(x) = \cos nx + \sin nx \qquad n = 0, \pm 1, \pm 2, \ldots$$

Proof not required

(a) Show that $\mathrm{CAS}_n(x)$ is orthogonal to $\mathrm{CAS}_{-n}(x)$ on the interval from $-\pi$ to π. Then $\mathrm{CAS}_n(x)$ is orthogonal to every $\mathrm{CAS}_k(x)$ for $k \neq n$. ← We know this from $\int \cos nx \cos kx = \int \cos nx \sin kx = \int \sin nx \sin kx = 0$

(b) Using orthogonality, find a formula for the kth coefficient h_k in a Hartley series:

$$f(x) = \sum_{n=-\infty}^{\infty} h_n \,\mathrm{CAS}_n(x) \qquad -\pi < x \le \pi$$

$\mathrm{CAS}_n(x)$ — THIS IS $\mathrm{CAS}_{-n}(x)$

(a) $$\int_{-\pi}^{\pi} (\cos nx + \sin nx)(\cos nx - \sin nx)\,dx$$

$$= \int_{-\pi}^{\pi} (\cos^2 nx - \sin^2 nx)\,dx = 0$$

THIS IS $\int \cos 2nx = 0$ OR each integral is π and $\pi - \pi = 0$.

(b) Multiply $f(x) = \sum h_n \mathrm{CAS}_n(x)$ by $\mathrm{CAS}_k(x)$ and INTEGRATE

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{CAS}_k(x)\,dx = \int_{-\pi}^{\pi} \sum h_n \mathrm{CAS}_n \mathrm{CAS}_k\,dx = h_k \int (\mathrm{CAS}_k)^2\,dx$$

ONLY 1 TERM IS NONZERO

We can find the number 2π

$$\int_{-\pi}^{\pi} (\mathrm{CAS})^2 = \underbrace{\int_{-\pi}^{\pi} \cos^2 kx}_{\to \pi} + \underbrace{2\int_{-\pi}^{\pi} \sin kx \cos kx}_{\to 0} + \underbrace{\int_{-\pi}^{\pi} \sin^2 kx}_{\to \pi}$$

$$\boxed{h_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{CAS}_k(x)\,dx}$$

(just like Fourier series but all real)

2. (35 points) The unit step function is $S(x) = \begin{cases} 1 \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases}$

(a) The Fourier integral transform of $S(x)$ is $\widehat{S}(k) = \dfrac{1}{ik} (k \neq 0)$. *Give a reason.* You could integrate the delta function $\delta(x)$ or you could let $a \to 0$ in a 1-sided pulse.

(b) To find the "step response" I want to solve $\boldsymbol{du/dx + au = S(x)}$ by Fourier transform for $-\infty < x < \infty$. Take the Fourier transform of each term to **find** $\boldsymbol{\widehat{u}(k)}$. To start the inverse transform, choose numbers C and D so that

$$\widehat{u}(k) = \frac{C}{ik} + \frac{D}{a+ik}. \quad = \frac{C(a+ik)+D(ik)}{(ik)(a+ik)}$$

(c) Take the inverse transform of both terms to find $u(x)$, remembering that the 1-sided pulse e^{-ax} transforms to $1/(a+ik)$ (Example 3 in Section 4.5). Put a box around your $u(x)$ and **check that** $\boldsymbol{du/dx + au = S(x)}$.

(d) (5 points) What is the decay rate of $\widehat{u}(k)$ and how can you see this from your $u(x)$?

(a) Why is $\hat{S}(k) = \frac{1}{ik}$? Reason 1 S = integral of δ Then $\hat{S} = \frac{1}{ik}\hat{\delta} = \frac{1}{ik}$

Reason 2  transforms to $\frac{1}{a+ik}$ Let $a \to 0$ / S transforms to $\frac{1}{0+ik}$

(b) $\frac{du}{dx} + au = S(x)$ $\rightarrow$ $\hat{u}(k) = \frac{1}{ik(a+ik)}$ = ABOVE

$ik\,\hat{u}(k) + a\,\hat{u}(k) = \frac{1}{ik}$

IF $C(a+ik) + D(ik) = 1$

THEN $C = 1/a$ $D = -1/a$

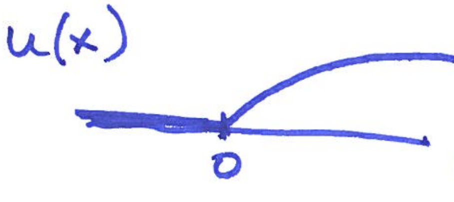
(c) Inverse transform of $\frac{1}{a\,ik} - \frac{1}{a(a+ik)}$ is $\boxed{\frac{1}{a}S(x) - \frac{1}{a}e^{-ax}}$

$$\boxed{u(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{a} - \frac{1}{a}e^{-ax} & x > 0 \end{cases}}$$

Check: $\frac{du}{dx} = \begin{cases} 0 & x < 0 \\ e^{-ax} & x \geq 0 \end{cases}$

Then $\frac{du}{dx} + au = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$ ✓

(d) Decay rate of $\hat{u}$ is $\frac{1}{k^2}$ $u(x)$ has a jump in slope



STEP RESPONSE WHEN A SWITCH TURNS ON

3. (25 points)

(a) Find the discrete Fourier transform (c_0, c_1, c_2, c_3) of the vector $f = (1, 0, 2, 0)$. Then $f = Fc$ for the Fourier matrix F. → NO PROOF REQUIRED

(b) I want to find a nonzero vector $d = (d_0, d_1, d_2, d_3)$ so that $\mathbf{c} \otimes \mathbf{d} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$. If $g = Fd$ use the convolution rule to turn $c \otimes d = (0, 0, 0, 0)$ into equations connecting $f = (1, 0, 2, 0)$ and $g = (g_0, g_1, g_2, g_3)$.

(c) Find a nonzero solution g to those equations. Then find $d = F^{-1}g$. Check that $c \otimes d = (0, 0, 0, 0)$.

(a)

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = F^{-1} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & -1 & - \\ 1 & - & 1 & - \\ 1 & - & -1 & - \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

(b) $c \circledast d = (0, 0, 0, 0)$ transforms to

$$\boxed{f .* g = 4 \begin{bmatrix} 1 \cdot g_0 \\ 0 \cdot g_1 \\ 2 \cdot g_2 \\ 0 \cdot g_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$f = Fc$

$g = Fd$

(c) You can choose any solution vector $g = \begin{bmatrix} 0 \\ g_1 \\ 0 \\ g_3 \end{bmatrix}$

If I choose $g = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ then $d = F^{-1}g =$

$$d = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

Check

$$c \circledast d = \frac{1}{4}(3, -1, 3, -1) \circledast \frac{1}{4}(2, 0, -2, 0)$$

$$= (0, 0, 0, 0)$$

6 − 6 etc.

4. (20 points)

(a) The vectors $u = (1, 1)$ and $v = (1, -1)$ are orthogonal vectors of length $\sqrt{2}$. For any vector $f = (f_0, f_1)$ find the numbers c_0 and c_1 so that $f = c_0 u + c_1 v$.

(b) Show that $\boldsymbol{f_0^2 + f_1^2 = 2(c_0^2 + c_1^2)}$ for every vector f: the energy identity.

Thank you for taking 18.085. I hope you enjoyed it—I enjoyed teaching you.

(a) $\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = c_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$f_0 = c_0 + c_1$

$f_1 = c_0 - c_1$

ADD / SUBTRACT

This is the DFT with $\underline{N = 2}$!

$c_0 = \dfrac{f_0 + f_1}{2} \quad c_1 = \dfrac{f_0 - f_1}{2}$

(b) $\underline{2(c_0^2 + c_1^2)} = 2\left(\dfrac{f_0^2 + 2f_0 f_1 + f_1^2}{4}\right) + 2\left(\dfrac{f_0^2 - \cancel{2f_0 f_1} + f_1^2}{4}\right)$

$= 4\dfrac{f_0^2}{4} + 4\dfrac{f_1^2}{4} = \underline{f_0^2 + f_1^2}$ ✓

THIS IS THE ENERGY IDENTITY FOR $N = 2$

The product $F\bar{F}^T$ is $2I$

$$\underset{F}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \underset{\bar{F}^T}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$$