Quiz III

18.085 (Dr. Christianson)
4/May/09

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Read the following:

- Do all your work on these pages. No calculators or computers may be used. Notes and the text may be used. The point value (out of 100) of each problem is marked in the margin. Note that some problems are worth more than others and plan your time accordingly. Please confine your answers to the given pages, and write your name on each page.
- The following may or may not be useful: there is a typo in the book on page 321, formula (15), $\frac{\pi}{4}$ should be $\frac{4}{\pi}$, as in formula (8) on page 318.

1. (*25 points*) This problem concerns Poisson's formula for the upper half-space in $\mathbb{R}^3$. Recall we denote a point in $\mathbb{R}^3$ by $\mathbf{x} = (x_1, x_2, x_3)$ or $\mathbf{y} = (y_1, y_2, y_3)$. The next two pages are blank for extra work.

a. (*5 pts*) The free Green's function for $\mathbb{R}^3$ is

$$G_0(\mathbf{x}, \mathbf{y}) = \frac{C}{\|\mathbf{x} - \mathbf{y}\|}$$

with C a constant you will determine. Verify that for any constant C,

$$-\Delta_{\mathbf{y}} G_0 = 0$$

except at $\mathbf{x} = \mathbf{y}$. What happens for $\mathbf{x} = \mathbf{y}$? (Recall $-\Delta_{\mathbf{y}} = -(\partial_{y_1}^2 + \partial_{y_2}^2 + \partial_{y_3}^2)$.)

b. (*10 pts*)
The constant C is chosen so that $-\Delta_{\mathbf{y}} G_0 = \delta(\mathbf{x} - \mathbf{y})$. For this choice of C, integrating over a ball of *any* radius $R > 0$ gives 1:

$$\int\int\int_{B(0,R)} (-\Delta_{\mathbf{y}} G_0(0, \mathbf{y})) dy_1 dy_2 dy_3 = \int\int\int_{B(0,R)} \delta(-\mathbf{y}) dy_1 dy_2 dy_3 = 1 .$$

by the definition of δ. Use the divergence theorem (or Gauss-Green formula) to conclude $C = \frac{1}{4\pi}$. (Hint: $G_0(0, \mathbf{y}) = C/\|\mathbf{y}\| = C/r$ in polar coordinates. Use this to show $\partial G_0/\partial n = dG_0/dr$ in polar coordinates, and hence $\partial G_0/\partial n$ is constant on a sphere of radius R.)

c. (*10 pts*)
In this part of the problem, we write $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$. Let $u(\mathbf{x})$ be the solution to the following BVP:

$$\begin{cases} -\Delta u(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \mathbb{R}^3_+, \\ u|_{\partial \mathbb{R}^3_+} = g. \end{cases}$$

Derive Poisson's formula for u:

$$u(\mathbf{x}) = \frac{x_3}{2\pi} \int\int \frac{g(y_1, y_2)}{(x_3^2 + (x_1 - y_1)^2 + (x_2 - y_2)^2)^{3/2}} dy_1 dy_2.$$

(Hint: $\partial \mathbb{R}^3_+ = \{(x_1, x_2, x_3) : x_3 = 0\}$ and reflection across the boundary is again given by $\tilde{\mathbf{x}} = (x_1, x_2, -x_3)$.)

a) $G_0(\vec{x},\vec{y}) = C\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{-1/2}$

$\partial_{y_1} G_0 = C(x_1-y_1)\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{-3/2}$

$$\partial_{y_1}^2 G_0 = C\Big[3(x_1-y_1)^2\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{-5/2}$$
$$-\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{-3/2}\Big]$$

similarly for $\partial_{y_2}^2$, $\partial_{y_3}^2$. Summing up:

$$(\partial_{y_1}^2+\partial_{y_2}^2+\partial_{y_3}^2)G_0 = C\Big[\frac{3(x_1-y_1)^2+3(x_2-y_2)^2+3(x_3-y_3)^2}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{5/2}}$$

$$-\frac{3}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)^{3/2}}\Big].$$ Multiplying the second term by $\dfrac{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)}{\left((x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2\right)}$ adds to zero.

If $\vec{x}\to\vec{y}$, G_0 is not continuous, so we cannot even differentiate it there.

b) $$1=\iiint_{B(0,R)} -\Delta_{\vec{y}} G_0(0,\vec{y})\,dy_1\,dy_2\,dy_3 = -\iint_{\partial B(0,R)} \frac{\partial G_0}{\partial n}\,dS$$

In polar coordinates, $G_0(0,\vec{y}) = \frac{C}{r}$, and $\hat{n}$ is in radial direction, so $\frac{\partial}{\partial n} = \nabla\cdot\hat{n} = \frac{\partial}{\partial r}$ in polar coordinates. Hence $\frac{\partial G_0}{\partial r}(0,r) = -\frac{C}{r^2}$

$$\Rightarrow -\iint_{\partial B(0,R)} \frac{\partial G_0}{\partial n}\,dS = \frac{C}{R^2}\iint_{\partial B(0,R)} dS = \frac{C}{R^2}(4\pi R^2)$$

$$\Rightarrow C = \frac{1}{4\pi}.$$

c) Green's function for $\mathbb{R}^3_+$ is $G(\vec{x},\vec{y}) = G_0(\vec{x},\vec{y}) - \phi^{\vec{x}}(\vec{y})$

where $\phi^{\vec{x}}(\vec{y})$ satisfies $\begin{cases} -\Delta_{\vec{y}}\, \phi^{\vec{x}}(\vec{y}) = 0 \text{ in } \mathbb{R}^3_+ \\ \phi^{\vec{x}}(\vec{y})\big|_{\partial\mathbb{R}^3_+} = G_0\big|_{\partial\mathbb{R}^3_+} \end{cases}$

$\phi^{\vec{x}}(\vec{y})$ is obtained by reflecting singularity at $\vec{y} = \vec{x}$ across the boundary : $\phi^{\vec{x}}(\vec{y}) = G_0(\tilde{\vec{x}},\vec{y})$ where $\tilde{\vec{x}} = (x_1, x_2, -x_3)$. A calculation as in Part (a) shows it satisfies $-\Delta_{\vec{y}}\, \phi^{\vec{x}}(\vec{y}) = 0$ in $\mathbb{R}^3_+$. Further,

$$\phi^{\vec{x}}(\vec{y})\Big|_{\partial\mathbb{R}^3_+} = G_0(\tilde{\vec{x}},\vec{y})\Big|_{\partial\mathbb{R}^3_+} = \frac{1}{4\pi}\left((x_1-y_1)^2 + (x_2-y_2)^2 + (-x_3-y_3)^2\right)^{-\frac{1}{2}}\Bigg|_{y_3=0}$$

$$= \frac{1}{4\pi}\left((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3)^2\right)^{-\frac{1}{2}}$$

$$= G_0\Big|_{\partial\mathbb{R}^3_+}.$$

For Poisson's formula, we just need $\frac{\partial G}{\partial n}$. $\hat{n}$ here is just $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, so $\frac{\partial}{\partial n} = \frac{-\partial}{\partial y_3}$, so

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial y_3}\Bigg|_{y_3=0} = \frac{1}{4\pi}\Big[-(x_3-y_3)\left((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2\right)^{-3/2}$$

$$+ (-x_3-y_3)\left((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2\right)^{-3/2}\Big]\Bigg|_{y_3=0}$$

$$= -\frac{x_3}{2\pi} \frac{1}{\left((x_1-y_1)^2 + (x_2-y_2)^2 + x_3^2\right)^{3/2}}$$

$$\Rightarrow u(\vec{x}) = -\iint_{\partial\mathbb{R}^3_+} \frac{\partial G}{\partial n} u \, dS = \frac{x_3}{2\pi} \iint_{\mathbb{R}^2} \frac{g(y_1,y_2)}{\left(x_3^2 + (x_1-y_1)^2 + (x_2-y_2)^2\right)^{3/2}}\, dy_1\, dy_2$$

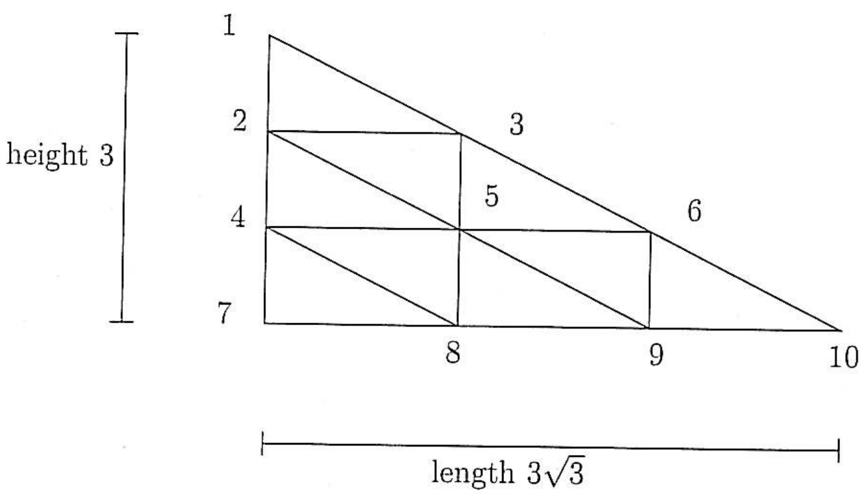

Figure 1: The triangle domain and mesh for problem 2.

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2. (*30 points*) This problem concerns the Finite Element Method for the triangular domain Ω in Figure 1. This is a right triangle, with bottom edge of length $3\sqrt{3}$ and left edge of length 3. Number the nodes as in the figure. We consider the following BVP:

$$\begin{cases} -\Delta u = 1 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

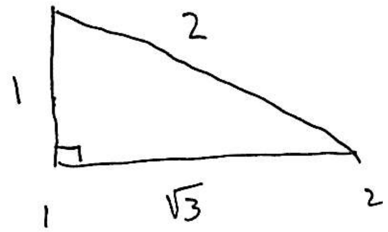
The next two pages are left blank for extra work.

a. (*10 pts*) Each small triangle in the mesh is the same size. Compute the 3×3 element matrix K_e for this problem for an individual triangle.

b. (*10 pts*) Compute the 5th row of the full element matrix K.

c. (*10 pts*)
Use the boundary conditions and the load function in (1) to get a reduced, non-singular matrix K_{red} and load vector $\mathbf{F}_{\text{red}}$ and solve for the interior values of $\mathbf{U}$, where $\mathbf{U}$ solves $K\mathbf{U} = \mathbf{F}$ with appropriate boundary conditions as usual.

3 a)



K_e is computed by putting the lower left corner at $(0,0)$. Then the position matrix becomes

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 1 & 0 & 1 \end{bmatrix}. \qquad C = P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$K_e = A\,\text{grad}^T\,\text{grad}$, where grad = last two rows of C

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

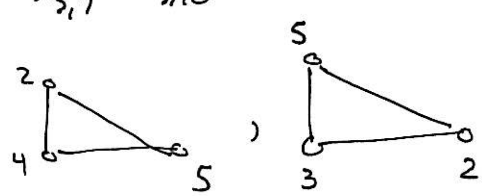
A = area

$$K_e = \frac{\sqrt{3}}{2}\begin{bmatrix} -1/\sqrt{3} & -1 \\ 1/\sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ -1 & 0 & 1 \end{bmatrix} = \frac{\sqrt{3}}{2}\begin{bmatrix} 4/3 & -1/3 & -1 \\ -1/3 & 1/3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(You can compute this in several ways, and it is only unique up to permuting the nodes.)

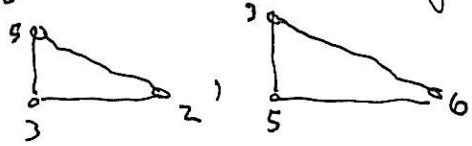
b) K_{5j} gets a contribution from each triangle with nodes 5 and j in it. Hence $K_{5,1} = K_{5,7} = K_{5,10} = 0$.

For K_{52}, there are two triangles:

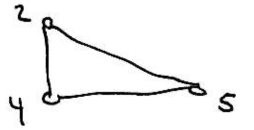


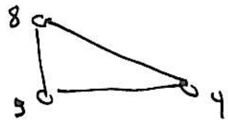
From K_e above, $K_{52} = 0 + 0 = 0$. Similarly, $K_{59} = 0$.

For K_{53}, 2 triangles



. Comparing to K_e gives $K_{53} = \frac{\sqrt{3}}{2}(-1-1) = -\sqrt{3}$. Similarly, $K_{58} = -\sqrt{3}$.

For K_{54}, 

, 

. Comparing to K_e

gives $K_{54} = \frac{\sqrt{3}}{2}\left(-\frac{1}{3}-\frac{1}{3}\right) = -\frac{1}{\sqrt{3}}$. Similarly, $K_{56} = -\frac{1}{\sqrt{3}}$.

For K_{55}, node 5 is in 6 triangles, twice in each of the three corners. Hence $K_{55} = 2\cdot\frac{\sqrt{3}}{2}\left(\frac{4}{3}+\frac{1}{3}+1\right) = \frac{8}{\sqrt{3}}$

Fifth row of $K = \begin{bmatrix} 0 & 0 & -\sqrt{3} & -\frac{1}{\sqrt{3}} & \frac{8}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & -\sqrt{3} & 0 & 0 \end{bmatrix}$

(quick check: row sums to zero? ✓)

c) Boundary conditions yield $K_{red} = \begin{bmatrix} 1 & & & & \\ & \ddots 1 & & & \\ & & \frac{8}{\sqrt{3}} & & \\ & & & 1 \ddots & \\ & & & & 1 \end{bmatrix}$ ← 5th row.

0 boundary conditions $\Rightarrow$ $\vec{F}_{red} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F_5 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$F_5 = \sum_{\substack{\text{triangles} \\ \text{with node} \\ 5}} \frac{(\text{area of triangle})}{3}\cdot 1$$

$$= 6\cdot\frac{\sqrt{3}}{6} = \sqrt{3}$$

Interior value of $\vec{U}$ is U_5, satisfying $\frac{8}{\sqrt{3}}U_5 = \sqrt{3}$, or

$$U_5 = \frac{3}{8}$$

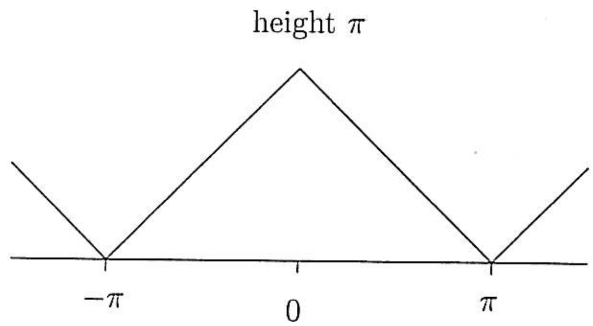


Figure 2: The periodic function $\mathrm{Hat}(x)$.

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3. (*20 points*) This problem concerns Fourier series for the 2π-periodic, piecewise linear function $\mathrm{Hat}(x)$ depicted in Figure 2. The next page is left blank for extra work.

a. (*10 pts*) Find the Fourier sin/cos series for $\mathrm{Hat}(x)$. (Hint: use any way you prefer).

b. (*5 pts*) Use your answer to part (a) to sum the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

c. (*5 pts*) Use your answer to part (a) to sum the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

(Hint: use Parseval's formula.)

3) a) $\text{Hat}(x) = \pi - RR(x)$, and $RR(x)$ has

F.S. $\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right]$.

Therefore, $\text{Hat}(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \cdots\right]$

$$= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}.$$

b) $\cos(0) = 1$, so $\text{Hat}(0) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$,

but also $\text{Hat}(0) = \pi$. Thus

$\pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, so $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$

c) By Parseval's formula,

$\int_{-\pi}^{\pi} |\text{Hat}(x)|^2 dx = 2\pi\left(\frac{\pi}{2}\right)^2 + \frac{16}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$, but also

$$\int_{-\pi}^{\pi} |\text{Hat}(x)|^2 dx = 2\int_0^{\pi} |\text{Hat}(x)|^2 dx = 2\int_0^{\pi}(\pi - x)^2 dx$$

$= -\frac{2}{3}(\pi - x)^3\Big|_0^{\pi} = \frac{2}{3}\pi^3$. Hence

$$\left(\frac{2}{3}\pi^3 - \frac{\pi^3}{2}\right)\frac{\pi}{16} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$$

$$\parallel$$

$$\frac{\pi^4}{96}$$

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4. (*25 points*) This problem concerns the wave equation for a string with free ends, modeled by the following I/BVP:

$$\begin{cases} u_{tt} = u_{xx}, \text{ for } 0 \le x \le \pi, \ t \ge 0, \\ u_x(0,t) = u_x(\pi,t) = 0, \\ u(x,0) = u_0, \ u_t(x,0) = u_1. \end{cases} \tag{2}$$

The next page is blank for extra work.

a. (*10 pts*) Use separation of variables to find the general solution to (2).

b. (*5 pts*) Use the Fourier series expansion of $\mathrm{Hat}(x)$ to find the particular solution for (2) with $u_0 = \mathrm{Hat}$ and $u_1 = 0$.

c. (*10 pts*) Find two functions, F_1 and F_2 (in terms of their Fourier series) so that

$$u(x,t) = F_1(x+t) + F_2(x-t)$$

(Hint: Use the trig identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$ to write your answer to part (b) in this form.) The Fourier series for F_1 and F_2 should look familiar. What are they? (This is called d'Alembert's solution to the wave equation. It shows that a solution to the wave equation is a sum of two waves, one travelling to the right and one travelling to the left.)

a) $u(x,t) = X(x)T(t) \rightarrow (T''X = X''T) \cdot \frac{1}{XT}$

$$\rightarrow \frac{T''}{T} = \frac{X''}{X} = \text{constant}$$

$\Rightarrow X'' = \lambda X$ eigen function equation.

$X'(0) = X'(\pi) = 0 \Rightarrow X(x) = \cos kx, \quad \lambda = -k^2$

$\Rightarrow T'' = -k^2 T$, so $T(t) = (a_k \cos kt + b_k \sin kt)$.

Summing up $\rightarrow u(x,t) = \sum_{k \geq 0} (a_k \cos kt + b_k \sin kt) \cos kx$.

b) Using Hat (x) from #3,

$$u(x,t) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{\substack{k \geq 1 \\ \text{odd}}} \frac{\cos kt \cos kx}{k^2}$$

$$= \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)t \cos(2n+1)x}{(2n+1)^2}$$

c) $\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\Rightarrow \cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B) \right)$

$$\Rightarrow u(x,t) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2} \left[\frac{\cos((2n+1)(t+x)) + \cos((2n+1)(t-x))}{(2n+1)^2} \right]$$

$$= \frac{1}{2}\left[\text{Hat}(x+t) + \text{Hat}(x-t) \right]$$