18.085 Homework #4

Problem.
MATLAB: In the middle of p. 155 is a fast construction of K= A'A when A is the incidence matrix for a square grid of nodes
B=toeplitz([2 -1 zeros(1,N-2)]), B(1,1)=1; B(N,N)=1;
K=kron(B, eye(N)) + kron(eye(N),B);
For N=3 then N=5 assign voltages to two corners u(1,1)=1; u(N,N)=0; and solve for all the other voltages Ku=(bring that voltage u(1,1) to the right side)
How much current flows out of node 1,1 and must reach node N,N? All unit resistances so C = I. Open question: How does the answer change with N?

>> N = 3

```
>> B=toeplitz([2 -1 zeros(1,N-2)]), B(1,1)=1; B(N,N)=1;
K=kron(B, eye(N)) + kron(eye(N),B);

>> K

K =

    2   -1    0   -1    0    0    0    0    0
   -1    3   -1    0   -1    0    0    0    0
    0   -1    2    0    0   -1    0    0    0
   -1    0    0    3   -1    0   -1    0    0
    0   -1    0   -1    4   -1    0   -1    0
    0    0   -1    0   -1    3    0    0   -1
    0    0    0   -1    0    0    2   -1    0
    0    0    0    0   -1    0   -1    3   -1
    0    0    0    0    0   -1    0   -1    2

>> K_red = K(2:8,2:8)

K_red =

    3   -1    0   -1    0    0    0
   -1    2    0    0   -1    0    0
    0    0    3   -1    0   -1    0
   -1    0   -1    4   -1    0   -1
    0   -1    0   -1    3    0    0
    0    0   -1    0    0    2   -1
    0    0    0   -1    0   -1    3

>> f = -K(2:8,1)

f =

    1
    0
    1
    0
    0
    0
    0
```

```
>> u = K_red\f

u =

   0.6667
   0.5000
   0.6667
   0.5000
   0.3333
   0.5000
   0.3333

>> u = [1; u; 0]

u =

   1.0000
   0.6667
   0.5000
   0.6667
   0.5000
   0.3333
   0.5000
   0.3333
        0
```

Same amount current flows out from node(1,1) to node(1,2) and node(2,1). Evenly distributed since all the resistance of edges are same.
And the at the same time, by Kirchhoff's current law, all the current flows out from node(1,1) have to flow in node(N,N)

Therefore the amount current flows out (when N =3) is

```
>> current_out =

   0.6667
```

```
clc
clear all

N = 3;
B=toeplitz([2 -1 zeros(1,N-2)]);
B(1,1)=1; B(N,N)=1;
K=kron(B, eye(N)) + kron(eye(N),B);

K_red = K(2:8,2:8);
f = -K(2:8,1);

u = K_red\f;
u = [1; u; 0]

current_out = 2*(1-u(2))
```

>> N = 5

u =

```
    1.0000
    0.7660
    0.6277
    0.5426
    0.5000
    0.7660
    0.6702
    0.5745
    0.5000
    0.4574
    0.6277
    0.5745
    0.5000
    0.4255
    0.3723
    0.5426
    0.5000
    0.4255
    0.3298
    0.2340
    0.5000
    0.4574
    0.3723
    0.2340
         0
```

current_out =

```
    0.4681
```

```
clc
clear all

N = 5;
B=toeplitz([2 -1 zeros(1,N-2)]);
B(1,1)=1; B(N,N)=1;
K=kron(B, eye(N)) + kron(eye(N),B);

K_red = K(2:24,2:24);
f = -K(2:24,1);

u = K_red\f;
u = [1; u; 0]

current_out = 2*(1-u(2))
```

How does the answer change with N?

As N increases, there are the more edges and nodes. This means within the fixed boundary condition that voltage applied at node(1,1) and ground at node(N,N) as N increases, total resistance is also increased. Therefore as N increases the total amount of current out from node(1,1) (same as current flows into node(N,N)) decreases.

18.085 Homework #4

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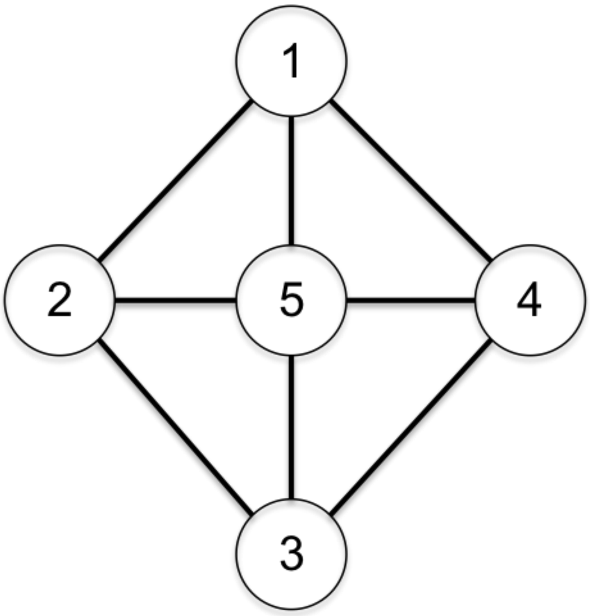
Problem #1

1A A network has N=4 nodes around a clock and 1 node at the center. 4 edges go around the clock and 4 in to the center.
The 12:00 node has voltage 1, the 6:00 node has voltage 0.
Edges around the clock have c=1, edges to the center have c=2
Set up and solve the network eqns with m=12 and n=5// and find the total current into the bottom 6:00 node
1B Change to N=12 nodes around the clock: n=13 and m=24.
1C Change to N=60 nodes around the clock: n=61 and m=120

1A

>> Without node at the center, nodes only around a clock can be expressed as "Circulant matrix"
If we add one more node at the center and connect it from outer nodes every node around a clock would have +2 in diagonal elements. And a row and a column is added to C matrix with -2s in off-diagonal and 2N at diagonal (N,N) component.




```
>>
clc
clear all

N = 4;
K = toeplitz([2 -1 zeros(1,N-2)]);
C = K; C(1,N)=-1; C(N,1)=-1;

K_new = [C; -2*ones(1,N)];
K_new = [K_new [-2*ones(1,N+1)]'];
K_new = K_new + 2*eye(N+1);
K_new(N+1,N+1) = 2*N


K_new =

     4    -1     0    -1    -2
    -1     4    -1     0    -2
     0    -1     4    -1    -2
    -1     0    -1     4    -2
    -2    -2    -2    -2     8
```

By setting the 12:00 node has voltage 1, the 6:00 node has voltage 0

```
node6 = N/2+1;
K_red = [K_new(2:node6-1,2:node6-1)     K_new(2:node6-1,node6+1:N+1);
         K_new(node6+1:N+1,2:node6-1)
K_new(node6+1:N+1,node6+1:N+1)];
```

With using reduced K and boundary condition solve equation

```
f = [-K_new(2:node6-1,1);
     -K_new(node6+1:N+1,1)];

u = K_red\f;
u_min = min(u);
u = [1; u; 0]
u_center = sum(u)/(N+1)

current = 2*(u_min) + 2*u_center
```

total current into the bottom 6:00 node current =

2

1B

```
clc
clear all

N = 12;
K = toeplitz([2 -1 zeros(1,N-2)]);
C = K; C(1,N)=-1; C(N,1)=-1;

K_new = [C; -2*ones(1,N)];
K_new = [K_new [-2*ones(1,N+1)]'];
K_new = K_new + 2*eye(N+1);
K_new(N+1,N+1) = 2*N;


node6 = N/2+1;
K_red = [K_new(2:node6-1,2:node6-1)     K_new(2:node6-1,node6+1:N+1);
         K_new(node6+1:N+1,2:node6-1)
K_new(node6+1:N+1,node6+1:N+1)];



f = [-K_new(2:node6-1,1);
     -K_new(node6+1:N+1,1)];

u = K_red\f;
u_min = min(u);
u = [1; u; 0]
u_center = sum(u)/(N+1)

current = 2*(u_min) + 2*u_center
```

u =

1.0000
0.6333
0.5333
0.5000
0.4667
0.3667
0.3667
0.4667
0.5000
0.5333
0.6333
0.5000
0

total current into the bottom 6:00 node current =

1.7333

1C

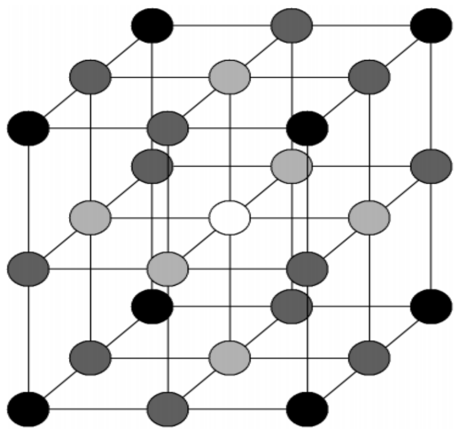
```
clc
clear all

N = 60;
K = toeplitz([2 -1 zeros(1,N-2)]);
C = K; C(1,N)=-1; C(N,1)=-1;

K_new = [C; -2*ones(1,N)];
K_new = [K_new [-2*ones(1,N+1)]'];
K_new = K_new + 2*eye(N+1);
K_new(N+1,N+1) = 2*N;


node6 = N/2+1;
K_red = [K_new(2:node6-1,2:node6-1)     K_new(2:node6-1,node6+1:N+1);
         K_new(node6+1:N+1,2:node6-1)
K_new(node6+1:N+1,node6+1:N+1)];



f = [-K_new(2:node6-1,1);
     -K_new(node6+1:N+1,1)];

u = K_red\f;
u_min = min(u);
u = [1; u; 0]
u_center = sum(u)/(N+1)

current = 2*(u_min) + 2*u_center
```

total current into the bottom 6:00 node current =

1.7321

Problem #2 Discussed with Geehoon and Sangha

a. For a 3D cubic grid with N^3 interior nodes and n=(N+2)^3 total nodes, describe the nxn singular matrix A'A and diag(A'A) for A=incidence matrix of this grid. What is the shape of A ? What is a typical inside row of A'A? Describe the boundary rows.
b. For N=2 create the reduced invertible 8x8 matrix K when all boundary values are ZERO. Display the pattern of nonzeros by spy(K). Invert K. Produce eig(K). Solve K*v=ones(8). Display v.



<Case when N=1>

a. 3D cubic grid is extended version of 2D graph that can describe how nodes are connected by edges. Therefore principles and rules of construction K = A'A are exactly same as for 2D graph.

Therefore,

- On the diagonal of A'A

$(A'A)_{jj}$ = degree = number of edges meeting at node j.

For a 3D cubic grid, there are four possible numbers which diagonal components can have: the numbers are 3, 4, 5, and 6

- 3: this is zero order of N, and exists at 8 corners in 3D cubic grid ➔ $8 \times N^0$
- 4: this is 1st order of N, exists on the 12 edges and N for each edge ➔ $12 \times N^1$
- 5: this is 2st order of N, exists on the 6 faces and N^2 for each face ➔ $6 \times N^2$
- 6: this is 3rd order of N, N^3 interior nodes ➔ N^3

➔ $N^3 + 6N^2 + 12^N + 8 = (N+2)^3 = n^3$

And also same as 2D graph, the columns still add to produce a zero column, therefore on each row there are -1s which cancels out 3 or 4 or 5 or 6 on the diagonal.

- Shape of A
  A = m x n

  1) Number of row, m is the number of edges
  We can calculate the number of edges based on logic used above
  Number of edges = $(3 \times 8 \times N^0 + 4 \times 12 \times N^1 + 5 \times 6 \times N^2 + 6 \times N^3)/2$

  2) Number of column, n is the number of nodes
  We can calculate the number of nodes
  Number of nodes = $(N+2)^3$

  Therefore we can define the shape of A as

$$(m \times n) \text{ rectangular matrix with}$$
$$m = (3 \times 8 \times N^0 + 4 \times 12 \times N^1 + 5 \times 6 \times N^2 + 6 \times N^3)/2$$
$$n = (N+2)^3$$

- Typical inside row of A'A
  [6 −1 −1 −1 −1 −1 −1 0 0 0 ⋯] with different arrangement of 6 and −1 while the 6 is on the diagonal and the number of −1s are always 6.

- Description of boundary rows
  There are 3 types of boundary rows
  1) 8 of [3 −1 −1 −1 0 0 0 ⋯] with 3 is on the diagonal and different arrangements of −1
  2) 12N of [4 −1 −1 −1 −1 0 0 ⋯] with 4 is on the diagonal and different arrangements of −1
  3) $6N^2$ of [5 −1 −1 −1 −1 −1 0 ⋯] with 5 is on the diagonal and different arrangements of −1

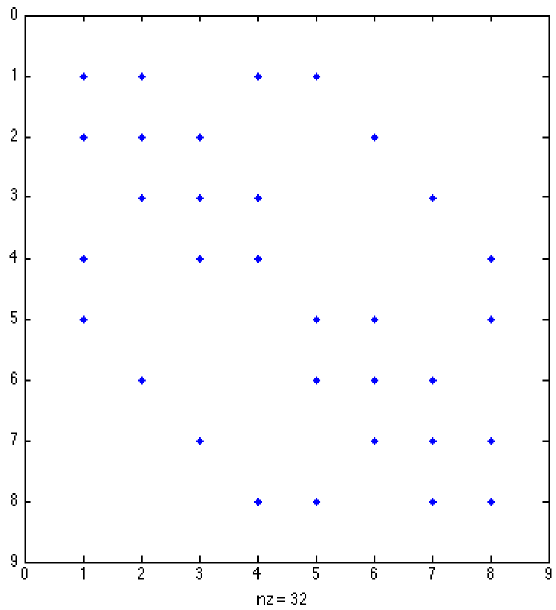
b.

K_red =

6	−1	0	−1	−1	0	0	0
−1	6	−1	0	0	−1	0	0
0	−1	6	−1	0	0	−1	0
−1	0	−1	6	0	0	0	−1
−1	0	0	0	6	−1	0	−1
0	−1	0	0	−1	6	−1	0

```
     0     0    -1     0     0    -1     6    -1
     0     0     0    -1    -1     0    -1     6
```

```
>> spy(K_red)
```




```
>> inv(K_red)

ans =

    0.1841    0.0349    0.0127    0.0349    0.0349    0.0127    0.0063
0.0127
    0.0349    0.1841    0.0349    0.0127    0.0127    0.0349    0.0127
0.0063
    0.0127    0.0349    0.1841    0.0349    0.0063    0.0127    0.0349
0.0127
    0.0349    0.0127    0.0349    0.1841    0.0127    0.0063    0.0127
0.0349
    0.0349    0.0127    0.0063    0.0127    0.1841    0.0349    0.0127
0.0349
    0.0127    0.0349    0.0127    0.0063    0.0349    0.1841    0.0349
0.0127
    0.0063    0.0127    0.0349    0.0127    0.0127    0.0349    0.1841
0.0349
    0.0127    0.0063    0.0127    0.0349    0.0349    0.0127    0.0349
0.1841

>> eig(K_red)
```

```
ans =

    3.0000
    5.0000
    5.0000
    5.0000
    7.0000
    7.0000
    7.0000
    9.0000
```

```
>> v = K_red₩ones(8)

v =

    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333    0.3333
0.3333
```