## SOLUTIONS FOR PSET 3

(2.1.1) Note that $\det(K) = c_1c_2c_3 + c_1c_2c_4 + c_1c_2c_4 + c_2c_3c_4$. In fixed-free case we set $c_4 = 0$ and get $\det(K) = c_1c_2c_3$. In free-free case we set $c_1 = c_4 = 0$ and get $\det(K) = 0$ so singular.

(2.1.4) In fixed-free case,

$$A^TCA = \begin{pmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}.$$

Adding three rows (or equivalently, multiplying $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ on the left), we get zeroes everywhere, so $f_1 + f_2 + f_3 = 0$.

Note that $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is a homogeneous solution, which is a solution of $A^TCAu = 0$. Also, $\begin{pmatrix} 1/c_2 & 0 & -1/c_3 \end{pmatrix}^T$ is a particular solution. Hence all solutions are $\begin{pmatrix} t + 1/c_2 & t & t - 1/c_3 \end{pmatrix}^T$ where $t$ ranges over all real numbers.

(2.1.7) Set $c_1 = c_3 = c_4 = 1$ and $c_2 = 0$. Then

$$K = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_3 & -c_3 \\ 0 & -c_2 & c_3 + c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

which is invertible since $\det(K) = 1$. Solving $Ku = f = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ we get $u = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}^T$.

Note that matrix $K$ is in the block-diagonal form, which means that mass 1 and mass 2, 3 are now in independent system (imagine the situation that we remove spring 2). So it is equivalent to solve each system independently.

(2.3.1) By calculus, one may want to find a solution of $E'(u) = 0$ which is a candidate for minimum. In fact $E''(u) = 2\sum_{i=1}^m a_i^2 \geq 0$ for any $u$, any solution of $E'(u) = 0$ gives a minimum. $E'(u) = 2(\sum_{i=1}^m a_i^2)u - 2\sum_{i=1}^m a_ib_i = 2A^TAu - 2A^Tb$ so $\hat{u} = \frac{A^Tb}{A^TA}$ which is exactly the solution of $A^TAu = A^Tb$.

(2.3.7) If the points were on a line, they will satisfy the equation $C + Dx = b$. So, we may set an equation

$$Au = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

which is not solvable. Instead, we find a solution for $A^TAu = A^Tb$. $A^TA = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}$ and $A^Tb = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ so $C = 3$ and $D = -1$.

(2.3.8) $p = \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix}^T$, $e = b - p = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}^T$. Check $A^Te = (0, 0)$.

(2.3.18) $\frac{9}{10}$, since $b_1 + \ldots + b_9 = 9\hat{u}_9$.

(2.3.24) Using the same technique as in (2.3.7), set

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

and solve $A^TAu = A^Tb$. We get $A^TA = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $A^Tb = (8, -3, -3)$ so $(C, D, E) = (2, -3/2, -3/2)$. Check that (average of $b$'s)$= 2 = C + 0 \cdot D + 0 \cdot E$.