

【 Problem 1.3-3】

1. Enter the matrix K_5 by the MATLAB command `toeplitz([2, -1, 0, 0, 0])`
2. Compute the determinant and the inverse by `det(K)` and `inv(K)`. For a neater answer compute the determinant times the inverse.
3. Find the L, D, U factors of K_5 and verify that the i, j entry of L^{-1} is j/i .

Solution:

`>> K5=toeplitz([2 -1 0 0 0])`

$K_5 =$

$$\begin{matrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{matrix}$$

`>> det(K5)`

$ans =$

6

`>> inv(K5)`

$ans =$

$$\begin{matrix} 0.8333 & 0.6667 & 0.5000 & 0.3333 & 0.1667 \\ 0.6667 & 1.3333 & 1.0000 & 0.6667 & 0.3333 \\ 0.5000 & 1.0000 & 1.5000 & 1.0000 & 0.5000 \\ 0.3333 & 0.6667 & 1.0000 & 1.3333 & 0.6667 \\ 0.1667 & 0.3333 & 0.5000 & 0.6667 & 0.8333 \end{matrix}$$

`>> det(K5)*inv(K5)`

$ans =$

$$\begin{matrix} 5.0000 & 4.0000 & 3.0000 & 2.0000 & 1.0000 \\ 4.0000 & 8.0000 & 6.0000 & 4.0000 & 2.0000 \\ 3.0000 & 6.0000 & 9.0000 & 6.0000 & 3.0000 \end{matrix}$$

2.0000	4.0000	6.0000	8.0000	4.0000
1.0000	2.0000	3.0000	4.0000	5.0000

>> [L,U] = lu(K5)

L =

1.0000	0	0	0	0
-0.5000	1.0000	0	0	0
0	-0.6667	1.0000	0	0
0	0	-0.7500	1.0000	0
0	0	0	-0.8000	1.0000

U =

2.0000	-1.0000	0	0	0
0	1.5000	-1.0000	0	0
0	0	1.3333	-1.0000	0
0	0	0	1.2500	-1.0000
0	0	0	0	1.2000

>> D=U*inv(L')

D =

2.0000	0	0	0	0
0	1.5000	0	-0.0000	-0.0000
0	0	1.3333	0	0
0	0	0	1.2500	0
0	0	0	0	1.2000

>> inv(L)

ans =

1.0000	0	0	0	0
0.5000	1.0000	0	0	0
0.3333	0.6667	1.0000	0	0
0.2500	0.5000	0.7500	1.0000	0
0.2000	0.4000	0.6000	0.8000	1.0000

It could be clearly verified that the i,j entry of inv(L) is j/i

【Problem 1.3-11】

The matrix $K = \text{ones}(4) + \text{eye}(4)/100$ has all 1's off the diagonal, and 1.01 down the main diagonal. Is it positive definite? Find the pivots by $\text{lu}(K)$ and eigenvalues by $\text{eig}(K)$. Also find its LDL^T factorization and $\text{inv}(K)$.

Solution:

It is positive definite.

Pivots are 1.01, 0.0199, 0.0150, 0.0133.

Eigenvalues are 0.01, 0.01, 0.01, 4.01

```
>> K= ones(4) + eye(4)/100
```

$K =$

1.0100	1.0000	1.0000	1.0000
1.0000	1.0100	1.0000	1.0000
1.0000	1.0000	1.0100	1.0000
1.0000	1.0000	1.0000	1.0100

```
>> lu(K)
```

$\text{ans} =$

1.0100	1.0000	1.0000	1.0000
0.9901	0.0199	0.0099	0.0099
0.9901	0.4975	0.0150	0.0050
0.9901	0.4975	0.3322	0.0133

```
>> eig(K)
```

$\text{ans} =$

0.0100
0.0100
0.0100
4.0100

```
>> [L,U]=lu(K)
```

$L =$

1.0000	0	0	0
0.9901	1.0000	0	0

0.9901	0.4975	1.0000	0
0.9901	0.4975	0.3322	1.0000

$U =$

1.0100	1.0000	1.0000	1.0000
0	0.0199	0.0099	0.0099
0	0	0.0150	0.0050
0	0	0	0.0133

$\gg D = U * \text{inv}(L')$

$D =$

1.0100	0	0	0
0	0.0199	0	-0.0000
0	0	0.0150	-0.0000
0	0	0	0.0133

$\gg \text{inv}(K)$

$\text{ans} =$

75.0623	-24.9377	-24.9377	-24.9377
-24.9377	75.0623	-24.9377	-24.9377
-24.9377	-24.9377	75.0623	-24.9377
-24.9377	-24.9377	-24.9377	75.0623

【Problem 1.3-17】

By trial and error, findn examples of 2×2 matrices such that

1. $LU \neq UL$
2. $A^2 = -I$, with real entries in A
3. $B^2 = 0$, with no zeros in B
4. $CD = -DC$, not allowing $CD = 0$

Solution:

$\gg L = [1, 0; 1, 1]$

$L =$

1	0
1	1

>> U=[1,1; 0,1]

U =

$$\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}$$

>> L*U

ans =

$$\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}$$

>> U*L

ans =

$$\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}$$

>> A=[0,1; -1, 0]

A =

$$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$$

>> A^2

ans =

$$\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$$

>> B=[1,1; -1, -1]

B =

$$\begin{matrix} 1 & 1 \\ -1 & -1 \end{matrix}$$

>> B^2

ans =

$$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

>> C=[0,1;1,0]

C =

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

>> D=[1,0; 0,-1]

D =

$$\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$$

>> C*D

ans =

$$\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$$

>> D*C

ans =

$$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$$

【Problem 1.4-2】

Change problem 1 to the free-fixed case $u'(0)=0$ and $u(1)=4$. Find and solve the four equations for A, B, C, D

Solution:

$u(x)$ must be linear on each side of the load position a and in the form:

$$u(x) = \begin{cases} Ax + B & \text{for } 0 \leq x \leq a \\ Cx + D & \text{for } a \leq x \leq 1 \end{cases}$$

The differential equation $-u'' = \delta(x - a)$ is conditioned by:
 $u'(0)=A=0$

$$u(1) = C + D = 4$$

$$u(a) = Aa + B = Ca + D$$

with the other integral equation at $x=a$: $\int_{a-}^{a+} (-u'') dx = 1 = -(C - A)$

we can solve these four equations for the coefficients A, B, C, D

$$\begin{cases} A = 0 \\ B = 5 - a \\ C = -1 \\ D = 5 \end{cases}$$



【Problem 1.4-6】

Show that $-u'' = \delta(x - a)$ with periodic conditions $u(0) = u(1)$ and $u'(0) = u'(1)$ cannot be solved. Again the requirements on C and D cannot be met. This corresponds to the singular circulant matrix C_n (with 1, n and n, 1 entries changed to -1)

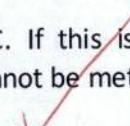
Solutions:

$-u'' = \delta(x - a)$ requires $u(x)$ to be linear on each side of the load position a .

So $u(x)$ is in the form

$$u(x) = \begin{cases} Ax + B & \text{for } 0 \leq x \leq a \\ Cx + D & \text{for } a \leq x \leq 1 \end{cases}$$

And $u(0) = u(1)$ and $u'(0) = u'(1)$ requires $B = D$ and $A = C$. If this is the scenario, the $-u''(x) = 0 \neq \delta(x - a)$. so the requirement on C and D cannot be met.



【Problem 1.4-7】

A difference of point loads, $f(x) = \delta\left(x - \frac{1}{3}\right) - \delta\left(x - \frac{2}{3}\right)$, does allow a free-free solution to $-u'' = f$. Find infinitely many solutions with $u'(0) = 0$ and $u'(1) = 0$

Solutions:

$U'' = 0$ at any position other than where the point loads locate. So $u(x)$ is linear at each side of the point load.

$$u(x) = \begin{cases} Ax + B & \text{for } 0 \leq x \leq \frac{1}{3} \\ Cx + D & \text{for } \frac{1}{3} \leq x \leq \frac{2}{3} \\ Ex + F & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}$$



$$\text{boundary conditions: } u\left(\frac{1}{3}-\right) = u\left(\frac{1}{3}+\right) \rightarrow \frac{A}{3} + B = \frac{C}{3} + D$$

$$u\left(\frac{2}{3}-\right) = u\left(\frac{2}{3}+\right) \rightarrow \frac{2C}{3} + D = \frac{2E}{3} + F$$

initial conditions:

$$u'(0) = 0 \rightarrow A = 0$$

$$u'(1) = 0 \rightarrow E = 0$$

$$\begin{cases} \int_{\frac{1}{3}}^{\frac{1}{3}+} (-u'') dx = 1 = -(C - A) \\ \int_{\frac{2}{3}-}^{\frac{2}{3}+} (-u'') dx = -1 = -(E - C) \end{cases} \rightarrow C = -1$$

Solve for these equations, we get the relations among these coefficients:

$$A = E = 0$$

$$C = -1$$

$$B = -\frac{1}{3} + D$$

$$F = -\frac{2}{3} + D$$

So we can set D to any real value for a solution:

$$u(x) = \begin{cases} -\frac{1}{3} + D & \text{for } 0 \leq x \leq \frac{1}{3} \\ -x + D & \text{for } \frac{1}{3} \leq x \leq \frac{2}{3} \\ -\frac{2}{3} + D & \text{for } \frac{2}{3} \leq x \leq 1 \end{cases}$$

so there are many solutions.



【Problem 1.4-12】

The cubic spline $C(x)$ solves the fourth order equation $u'''' = \delta(x)$. what is the complete solution $u(x)$ with four arbitrary constants? Choose those constants so that $u(1) = u''(1) = u(-1) = u''(-1) = 0$. This gives the bending of a uniform simply supported beam under a point load/.

Solutions:

$$u'''' = \delta(x)$$

$$\rightarrow u''' = s(x) + A \quad (s(x) \text{ is step function})$$

$$\rightarrow u'' = \text{ramp}(x) + Ax + B$$

$$\rightarrow u' = \begin{cases} \frac{A}{2}x^2 + Bx + C & \text{for } x \leq 0 \\ \left(\frac{1}{2} + \frac{A}{2}\right)x^2 + Bx + C & \text{for } x \geq 0 \end{cases}$$

$$\rightarrow u = \begin{cases} \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D & \text{for } x \leq 0 \\ \left(\frac{1}{6} + \frac{A}{6}\right)x^3 + \frac{B}{2}x^2 + Cx + D & \text{for } x \geq 0 \end{cases}$$

The complete solution u is given above.

$$\text{Given conditions: } \begin{cases} u(1) = 0 \\ u''(1) = 0 \\ u(-1) = 0 \\ u''(-1) = 0 \end{cases} \rightarrow \begin{cases} \frac{A+1}{6} + \frac{B}{2} + C + D = 0 \\ (1+A) + B = 0 \\ -\frac{A}{6} + \frac{B}{2} - C + D = 0 \\ -A + B = 0 \end{cases} \rightarrow \begin{cases} A = -\frac{1}{2} \\ B = -\frac{1}{2} \\ C = 0 \\ D = \frac{1}{6} \end{cases}$$

【Problem 1.5-1】

The 2×2 matrix K_2 in example 1 has eigenvalues 1 and 3 in Λ . Its unit eigenvectors q_1 and q_2 are the columns of Q . multiply $Q \wedge Q^T$ to recover K_2 .

Solutions:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Q \wedge Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = K_2$$

【Problem 1.5-2】

When you multiply the eigenvector $y = (\sin \pi h, \sin 2\pi h \dots)$ by K , the first row will produce a multiple of $\sin \pi h$. Find the multiplier λ by a double-angle formula for $\sin 2\pi h$:

$(Ky)_1 = 2\sin \pi h - \sin 2\pi h = \lambda \sin \pi h$ then $\lambda = \underline{\hspace{2cm}}$

Solutions:

$$2\sin \pi h - \sin 2\pi h = 2 \sin \pi h * (1 - \cos \pi h) = 2 \sin \pi h * 2 \sin^2 \frac{\pi h}{2}$$

$$\lambda = 4 \sin^2 \frac{\pi h}{2}$$

【Problem 1.5-3】

In matlab, construct $K=K_5$ and then its eigenvalues by $e=\text{eig}(K)$. that column should be

$(2 - \sqrt{3}, 2 - 1, 2 - 0, 2 + 1, 2 + \sqrt{3})$. Verify that e agrees with
 $2*\text{ones}(5,1)-2*\cos([1:5]*\pi/6)'$.

Solutions:

```
>> K5=2*eye(5)-diag(ones(4,1),1)-diag(ones(4,1),-1)
```

$K5 =$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

>> e=eig(K5)

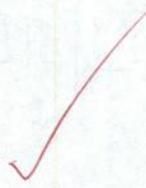
e =

$$\begin{bmatrix} 0.2679 \\ 1.0000 \\ 2.0000 \\ 3.0000 \\ 3.7321 \end{bmatrix}$$

>> 2*ones(5,1)-2*cos([1:5]*pi/6)'

ans =

$$\begin{bmatrix} 0.2679 \\ 1.0000 \\ 2.0000 \\ 3.0000 \\ 3.7321 \end{bmatrix}$$



【Problem 1.6-9】

For which numbers b and c are these matrices positive definite?

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$$

With the pivots in D and multiplier in L, factor each A into LDL^T .

Solutions:

$$\text{Case 1: } A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 - 10\lambda + 9 - b^2 \\ \Delta &= 10^2 - 4 \times (9 - b^2) = 4b^2 + 64 > 0 \end{aligned}$$

And the function $f(\lambda) = \lambda^2 - 10\lambda + 9 - b^2$ is centered at $\lambda = 5$. For it to have 2 positive roots, $f(0) > 0 \rightarrow -3 < b < 3$

$$A = LDL^T = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$



Case 2: $A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$

$$\det(A - \lambda I) = \lambda^2 - (2+c)\lambda + 2c - 16$$

$$\Delta = (2+c)^2 - 4 \times (2c - 16) = c^2 - 4c + 68 = (c-2)^2 + 64 > 0$$

And the function $f(\lambda) = \lambda^2 - (2+c)\lambda + 2c - 16$ is centered at $\lambda = 1 + \frac{c}{2}$. For it to

have 2 positive roots, $\begin{cases} f(0) > 0 \\ 1 + \frac{c}{2} > 0 \end{cases} \rightarrow \begin{cases} c > 8 \\ c > -2 \end{cases} \rightarrow c > 8$

$$A = LDL^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

【Problem 1.6-12】

Test the columns of A to see if $A^T A$ will be positive definite in each case:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solutions:

$K = A^T A$ is symmetric positive definite if and only if A has independent columns.

For $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, columns of A are independent. So $A^T A$ will be positive definite.

For $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$, columns of A are independent. So $A^T A$ will be positive definite.

For $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, columns of A are dependent. So $A^T A$ will not be positive definite.

【Problem 1.6-20】

Without multiplying $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, find

- (a) The determinant of A
- (b) The eigenvalues of A
- (c) The eigenvectors of A
- (d) A reason why A is symmetric positive definite.

Solutions:

(a) $\det(A) = \det\left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}\right) * \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\right) * \det\left(\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}\right) = 1 * (2 * 5) * 1 = 10$

(b) $\lambda_1 = 2, \lambda_2 = 5$

$$(c) \quad u_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad u_2 = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

(d) $A = Q \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} Q^T$, where the matrix in the middle is positive definite. The

orthogonal matrix Q is invertible. All eigenvalues in $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ are positive.