

【Problem 1.1-1】

$$\text{Guess } T_5^{-1} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \checkmark$$

$$T_5 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$T_5^{-1} \times T_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5$$

Simple formula for the entries of $T_n^{-1}(i,j) = \begin{cases} n-i+1, & i \geq j \\ n-j+1, & i \leq j \end{cases}$

【Problem 1.1-2】

Step1: check that $T_3 = U^T U$

$$U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad U^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$U^T \times U = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \checkmark$$

Step2: check that $U \times U^{-1} = I$

$$U^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$U^{-1} \times U = I_3$$

Step3: Invert $U^T U$ to find $T_3^{-1} = (U^{-1})(U^{-1})^T$

$$T_3^{-1} = (U^T U)^{-1} = (U^{-1})(U^T)^{-1} = (U^{-1})(U^{-1})^T \checkmark$$

The last equality above is because $(U^T)^{-1} = (U^{-1})^T$,

(Cause $U^T \times (U^{-1})^T = (U^{-1} \times U)^T = I^T = I$)

【Problem 1.1-15】

How many individual multiplications to create Ax and A^2 and AB

Expression	Multiplications
$A_{n \times n} x_{n \times 1}$	$N * N \checkmark$
$A_{n \times n} A_{n \times n}$	$N * N * N \checkmark$
$A_{m \times n} B_{n \times p}$	$M * N * P \checkmark$

【Problem 1.1-27】

Show that the 3 by 3 matrix K comes from $A_0 A_0'$:

$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ is a "difference matrix"}$$

Which column of A_0 would you remove to produce A_1 with $T = A_1 A_1'$? Which column would you remove next to produce A_2 with $B = A_2 A_2'$? The difference matrices A_0 , A_1 , A_2 have 0, 1, 2 boundary conditions. So do the "second differences" K , T , and B .

$$A_0 A_0' = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = K_3$$

$$A_1 = A_0 \text{ with the 4th column removed} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = -U$$

(matrix U is the same as in problem 1.1 - 15)

$$\text{So } A_1 A_1' = (-U)(-U)' = U U' = T_3$$

$$A A' = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \neq T$$

$$A_2 = A_1 \text{ with the 1st column removed} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\text{So } A_2 A_2' = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = B_3 \quad \checkmark$$

【Problem 1.2-1】

What are the second derivative $u''(x)$ and the second difference $\Delta^2 U_n$? Use $\delta(x)$

$$u(x) = \begin{cases} Ax & \text{if } x \leq 0 \\ Bx & \text{if } x \geq 0 \end{cases} \quad U_n = \begin{cases} An & \text{if } n \geq 0 \\ Bn & \text{if } n \leq 0 \end{cases} = \begin{bmatrix} -2A \\ -A \\ 0 \\ B \\ 2B \end{bmatrix}$$

Solution:

$$\text{Clearly, } u'(x) = \begin{cases} A & \text{if } x < 0 \\ B & \text{if } x > 0 \end{cases} \quad \text{and} \quad u''(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

For $u''(x)$ at $x=0$, we calculate the integration:

$$\int_{0^-}^{0^+} u''(x) dx = u'(x)|_{0^-}^{0^+} = u'(0^+) - u'(0^-) = B - A$$

$$\text{So } u''(x) = (B - A)\delta(x) \quad \checkmark$$

$$\text{So } u''(x) = \begin{cases} 0 & \text{if } x < 0 \\ (B - A)\delta(x) & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

As for U_n , the second difference of a linear part is zero. So the only non-zero can only appear at the 3rd line. Its value equals $B - 2 \cdot 0 + (-A) = B - A$

$$\Delta^2 U_n = \begin{bmatrix} 0 \\ 0 \\ B - A \\ 0 \\ 0 \end{bmatrix}$$

【Problem 1.2-2】

Solve the differential equation $-u''(x) = \delta(x)$ with $u(-2)=0$ and $u(3) = 0$. The pieces $u=A(x+2)$ and $u=B(x-3)$ meet at $x=0$. Show that the vector $U=(u(-1), u(0), u(1), u(2))$ solves the corresponding matrix problem $KU = F = (0, 1, 0, 0)$.

Solution:

$-u''(x) = 0$ when x is not zero.

$$\text{So } u(x) = u(x) = \begin{cases} Ax + C & \text{if } x < 0 \\ Bx + D & \text{if } x > 0 \end{cases}$$

$$\int_{0^-}^{0^+} u''(x) dx = u'(x)|_{0^-}^{0^+} = u'(0^+) - u'(0^-) = B - A = \int_{0^-}^{0^+} \delta(x) dx = 1$$

So we have $B - A = 1$

$$u(-2) = -2A + C = 0$$

$$u(3) = 3B + D = 0$$

and at $x=0$, pieces $u=A(x+2)$ and $u=B(x-3)$ meet, which gives us: $2A = -3B$

we can solve for coefficients A, B, C, D :

$$A = -3/5, \quad B = 2/5, \quad C = -6/5, \quad D = -6/5$$

$$u(x) = \begin{cases} -3(x+2)/5 & \text{if } x \leq 0 \\ 2(x-3)/5 & \text{if } x \geq 0 \end{cases}$$

$$U = (u(-1), u(0), u(1), u(2)) = \left(-\frac{3}{5}, -\frac{6}{5}, -\frac{4}{5}, -\frac{2}{5}\right)$$

$$\text{So } KU = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} \\ -\frac{6}{5} \\ -\frac{4}{5} \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus U solves the corresponding matrix problem $KU = F = (0, 1, 0, 0)$

【Problem 1.2-6】

For $u(x)=x^4$, compute the second derivative and second difference $\frac{\Delta^2 u}{(\Delta x)^2}$. From the answers, predict c in the leading error in equation(9)

Solution:

Second derivative of $u(x)=x^4$ is $u''(x) = (x^4)'' = 12x^2$

Second difference

$$\begin{aligned} \frac{\Delta^2 u}{(\Delta x)^2} &= \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2} \xrightarrow{\Delta x=h} \frac{(x+h)^4 - 2x^4 + (x-h)^4}{h^2} \\ &= \frac{12x^2 h^2 + 2h^4}{h^2} = 12x^2 + 2h^2 = u''(x) + ch^2 u''''(x) \end{aligned}$$

And we have $u''''(x)=24$

Compare this to equation(9), leading error $c=1/12$ ✓

【Problem 1.2-9】

Show that the 4th difference $\frac{\Delta^4 u}{(\Delta x)^4}$ with coefficients 1, -4, 6, -4, 1 approximates $\frac{d^4 u}{dx^4}$

by testing on $u = x, x^2, x^3, x^4$:

$$\frac{\Delta^4 u}{(\Delta x)^4} = \frac{u_2 - 4u_1 + 6u_0 - 4u_{-1} + u_{-2}}{(\Delta x)^4} = \frac{d^4 u}{dx^4} + (\text{which leading error?})$$

Solution:

let $\Delta x = h$

when $u=x$,

$$\frac{d^4 u}{dx^4} = 0,$$

$$\frac{\Delta^4 u}{(\Delta x)^4} = \frac{u_2 - 4u_1 + 6u_0 - 4u_{-1} + u_{-2}}{(\Delta x)^4} = \frac{2h - 4h + 6 \cdot 0 - 4(-h) + (-2h)}{h^4} = 0 = \frac{d^4 u}{dx^4} \quad \text{leading error}=0 \quad \checkmark$$

When $u=x^2$,

$$\frac{d^4 u}{dx^4} = 0, \quad \frac{\Delta^4 u}{(\Delta x)^4} = \frac{(2h)^2 - 4h^2 + 6 \cdot 0 - 4(-h)^2 + (-2h)^2}{h^4} = 0 = \frac{d^4 u}{dx^4} \quad \text{leading error}=0 \quad \checkmark$$

When $u=x^3$,

$$\frac{d^4 u}{dx^4} = 0, \quad \frac{\Delta^4 u}{(\Delta x)^4} = \frac{(2h)^3 - 4h^3 + 6 \cdot 0 - 4(-h)^3 + (-2h)^3}{h^4} = 0 = \frac{d^4 u}{dx^4} \quad \text{leading error}=0 \quad \checkmark$$

When $u=x^4$,

$$\frac{d^4 u}{dx^4} = 24, \quad \frac{\Delta^4 u}{(\Delta x)^4} = \frac{(2h)^4 - 4h^4 + 6 \cdot 0 - 4(-h)^4 + (-2h)^4}{h^4} = 24 = \frac{d^4 u}{dx^4} \quad \text{leading error}=0 \quad \checkmark$$

As can be seen, the 4th difference $\frac{\Delta^4 u}{(\Delta x)^4}$ with coefficients 1, -4, 6, -4, 1 approximates

$\frac{d^4 u}{dx^4}$ at least to the 4th power of x .

For general u , $\frac{\Delta^4 u}{(\Delta x)^4} \approx \frac{d^4 u}{dx^4} + c(\Delta x)^2 u^{(6)}$.

【Matlab Problem 1.1-3】

>> U=eye(5)-diag(ones(4,1),1)

U =

```
1 -1 0 0 0
0 1 -1 0 0
0 0 1 -1 0
0 0 0 1 -1
0 0 0 0 1
```

>> S=triu(ones(5))

S =

```
1 1 1 1 1
0 1 1 1 1
0 0 1 1 1
0 0 0 1 1
0 0 0 0 1
```

>> U*S

ans =

```
1 0 0 0 0
0 1 0 0 0
0 0 1 0 0
0 0 0 1 0
0 0 0 0 1
```

【Matlab Problem 1.1-5】

Guess $K_5 = \frac{1}{6} \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$, guess $\det(K_5) = \frac{1}{6}$, $\det(K_2) = \frac{1}{3}$

>> K5=2*eye(5)-diag(ones(4,1),1)-diag(ones(4,1),-1)

K5 =

```

    2   -1   0   0   0
   -1    2  -1   0   0
    0   -1   2  -1   0
    0    0  -1   2  -1
    0    0   0  -1   2

```

```
>> det(K5)
```

```
ans =
```

```
6
```

```
>> inv(K5)
```

```
ans =
```

```

    0.8333    0.6667    0.5000    0.3333    0.1667
    0.6667    1.3333    1.0000    0.6667    0.3333
    0.5000    1.0000    1.5000    1.0000    0.5000
    0.3333    0.6667    1.0000    1.3333    0.6667
    0.1667    0.3333    0.5000    0.6667    0.8333

```

```
>> det(K5)*inv(K5)
```

```
ans =
```

```

    5.0000    4.0000    3.0000    2.0000    1.0000
    4.0000    8.0000    6.0000    4.0000    2.0000
    3.0000    6.0000    9.0000    6.0000    3.0000
    2.0000    4.0000    6.0000    8.0000    4.0000
    1.0000    2.0000    3.0000    4.0000    5.0000

```

【 Matlab Problem 1.2-19 】

```

>> h=1/5;
K4=2*eye(4)-diag(ones(3,1),1)-diag(ones(3,1),-1);
deltaP= -eye(4)+diag(ones(3,1),1);
deltaN= eye(4) -diag(ones(3,1),-1);
delta0=(deltaP+deltaN)/2;
(K4/h^2+delta0/2/h)          %centered
f=[1,1,1,1]';
(K4/h^2+delta0/2/h)\f

```

```
ans =
```

```

50.0000 -23.7500      0      0
-26.2500  50.0000 -23.7500      0
      0 -26.2500  50.0000 -23.7500
      0      0 -26.2500  50.0000

```

ans =

```

0.0758
0.1175
0.1215
0.0838

```

>>

```

U=eye(4)-diag(ones(3,1),1);
(K4/h^2+deltaP*U/h) %uncentered
(K4/h^2+deltaP*U/h)\f

```

ans =

```

45.0000 -15.0000 -5.0000      0
-25.0000  45.0000 -15.0000 -5.0000
      0 -25.0000  45.0000 -15.0000
      0      0 -25.0000  45.0000

```

ans =

```

0.0699
0.1066
0.1090
0.0828

```

When $x=0$, $u(x) = u(0) = 0 + A + B = 0$.

when $x=1$, $u(x) = u(1) = 1 + A + Be = 0$.

Solve for A, B, we get $\begin{cases} A = \frac{1}{e-1} \\ B = \frac{1}{1-e} \end{cases}$, Satisfying the boundary conditions.

$$u(x) = x + \frac{1}{e-1} + \frac{1}{1-e} e^x$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0.0711 \\ 0.1138 \\ 0.1215 \\ 0.0868 \end{pmatrix}$$

more close to u
than U ✓

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