18.085 SUMMER 2014 - QUIZ 3 - AUGUST 15, 2014

YOUR NAME: ______________________________

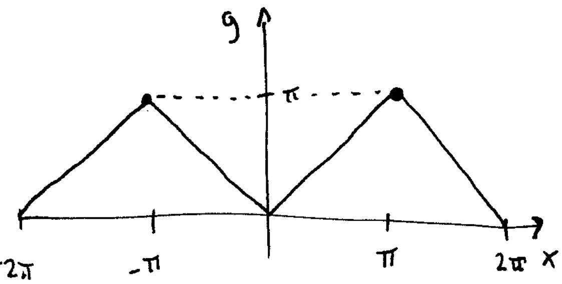
YOUR SCORE: ____________ **/ 100 +** ____________ **BONUS**

This exam has 5 questions!

Question 1 (5+8+3+3+6=25pts) Real and complex Fourier series.

Let $f(x) = |x|$ be defined on $-\pi \leq x < \pi$, and let $g(x)$ be its 2π-periodic extension.

(a) Sketch the function $g(x)$. Label its maximum value(s), and the location(s) of the maximum value(s), on your graph.



(b) Compute the real Fourier coefficients of $g(x)$. You may use the symmetry of the function to reduce your workload, and use integration by parts.

g is even : $b_n = 0$ for all n

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$\int u\,dv = uv - \int v\,du$

$u = x$ $\quad du = dx$

$v = \frac{\sin nx}{n}$ $\quad dv = \cos nx\,dx$

(integration by parts) $$= \frac{2}{\pi}\left[\frac{x\sin nx}{n}\Big|_0^{\pi} - \int \frac{\sin nx\,dx}{n}\right]$$

$$= \frac{2}{\pi}\left[0 - 0 - (-1)\frac{\cos nx}{n^2}\Big|_0^{\pi}\right] = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

so
$\boxed{a_n = 0 \text{ for } n \text{ even}}$
$\boxed{a_n = \frac{-4}{\pi n^2} \text{ for } n \text{ odd}}$
$\boxed{a_0 = \frac{\pi}{2}}$

(c) What are the complex Fourier coefficients of $g(x)$? You may use your answer from part (b) to reduce your workload.

We know the rules $a_0 = c_0$, $a_k = c_k + c_{-k}$ and $b_k = c_k - c_{-k}$

Since $b_k = 0$, we get $c_k = c_{-k}$. This means $a_k = c_k + c_{-k} = 2c_k$,

or $c_k = \frac{1}{2} a_k = \frac{-2}{k^2}$. And $c_0 = a_0 = \pi/2$.

We conclude: $\boxed{c_0 = \pi/2}$

$\boxed{c_k = -2/k^2 \text{ for } k \text{ odd} \quad (c_k = 0 \text{ for } k \text{ even})}$

(d) How fast do the Fourier coefficients of $g(x)$ decay as $n \to \infty$? Explain why you could have predicted this without doing parts (b) or (c).

we have a decay of $\frac{1}{n^2}$, which we expect from a continuous function whose 1st derivative is discontinuous

(e) Use Parseval's theorem to deduce the following equality:

$$\sum_{n\geq 1,\text{ odd}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$$

Parseval: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx = a_0^2 + \frac{1}{2}\sum_{n\,odd} a_n^2$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx = \frac{2}{2\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left.\frac{x^3}{3}\right|_0^{\pi} = \frac{\pi^3}{3\pi} - 0 = \frac{\pi^2}{3}$$

$$a_0^2 + \frac{1}{2}\sum_{n\,odd} a_n^2 = \frac{\pi^2}{4} + \frac{1}{2}\sum_{n\,odd}\frac{16}{\pi^2 n^4} = \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{n\,odd}\frac{1}{n^4}$$

Putting things together: $\frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{8}{\pi^2}\sum_{odd}\frac{1}{n^4}$

$$\Downarrow$$

$$\frac{\pi^2}{12} = \frac{8}{\pi^2}\sum_{n\,odd}\frac{1}{n^4}$$

or $$\frac{\pi^4}{96} = \sum_{n\,odd}\frac{1}{n^4}$$

Question 2 (3+3+4+6=16 pts) The Discrete Fourier Transform Part 1.

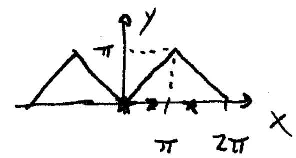
Let $w = e^{i2\pi/N}$.

(a) The entries of the Fourier matrix F are $F_{jk} =$ w^{jk}.

(b) The entries of the inverse of the Fourier matrix F^{-1} are $F^{-1}_{jk} =$ $\frac{1}{N}\bar{w}^{jk}$.

Let $y(x)$ be the following function (assumed to be 2π-periodic):

$$y(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi. \end{cases}$$



(c) Sample $y(x)$ at the $N = 3$ Fourier sample points to get the sample vector $\vec{y} = (y(0), \cdots)$:

$$\vec{y} = (0,\ 2\pi/3,\ 2\pi/3)^T.$$

$y(0),\ y\left(\frac{2\pi}{3}\right),\ y\left(2\cdot\frac{2\pi}{3}\right)$

$0 \qquad 2\pi/3 \qquad 2\pi/3$

(d) Find the discrete Fourier coefficients $\vec{c}$ of $\vec{y}$ by solving the equation $\vec{y} = F\vec{c}$, again for $N = 3$. The answer uses numbers, no w's and no square roots! You might want to recall that columns and rows of F sum to 0, except for the 0th row and column.

$\vec{y} = F\vec{c}, \quad \vec{c} = F^{-1}\vec{y}$

$$\vec{c} = \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{w} & \bar{w}^2 \\ 1 & \bar{w}^2 & \bar{w}^4 \end{pmatrix}\begin{pmatrix} 0 \\ 2\pi/3 \\ 2\pi/3 \end{pmatrix} \longrightarrow$$

where $\bar{w} = w^{-1} = w^2$ for $N=3$

(since $w^3 = 1$, $w^{-1}w^3 = w^{-1}\cdot 1$ but also $= w^2$)

and $\bar{w}^2 = w^{-2} = w^1$

and $\bar{w}^4 = w^{-4} = w^{-1} = w^2$

$$= \frac{1}{N}\begin{pmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{pmatrix}\begin{pmatrix} 0 \\ 2\pi/3 \\ 2\pi/3 \end{pmatrix}$$

row 1 of F

$$= \frac{1}{3}\begin{pmatrix} 4\pi/3 \\ 2\pi/3\,(w^2+w) \\ 2\pi/3\,(w^2+w) \end{pmatrix} \rightarrow$$

but we know $1 + w + w^2 = 0$ (when $N=3$)

so $w + w^2 = -1$

$$= \frac{1}{3}\,\frac{2\pi}{3}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \boxed{\frac{2\pi}{9}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \vec{c}}$$

Question 3 (6+6+4=16 pts) The Discrete Fourier Transform Part 2.

You are given the vector $\vec{c} = \frac{1}{3}(2,-1,-1)^T$ (different from part 1).

(a) Take the cyclic convolution of the vector $\vec{c}$ with itself: $\vec{c} \circledast \vec{c}$. You can use the direct method you like ("long multiplication" or using a circulant matrix), but DO NOT use the convolution rule.

$$\frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4+1+1 \\ -2-2+1 \\ -2+1-2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} \quad \leftarrow \text{using circulant matrix}$$

OR

$$\frac{1}{9}(\underset{w^0}{2}, \underset{w^1}{-1}, \underset{w^2}{-1}) \circledast (\underset{w^0}{2}, \underset{w^1}{-1}, \underset{w^2}{-1}) = \frac{1}{9}(\underset{w^0,\ w^3}{4+1+1},\ \underset{w^1,\ w^4}{-2-2+1},\ \underset{w^2}{-2-2+1}) = \frac{1}{9}(6,-3,-3)$$

(b) Multiply the Fourier matrix F (for $N = 3$) with the convolution of (a). That is, calculate $F(\vec{c} \circledast \vec{c})$. The answer uses numbers, not w's.

$$F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}, \quad F(\vec{c} \circledast \vec{c}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} \cdot \frac{1}{9} = \frac{1}{9}\begin{pmatrix} 6-3-3 \\ 6-3(w+w^2) \\ 6-3(w+w^2) \end{pmatrix}$$

$$F(\vec{c} \circledast \vec{c}) = \frac{1}{9}\begin{pmatrix} 0 \\ 6+3 \\ 6+3 \end{pmatrix} \leftarrow \text{from } 1+w+w^2=0 \text{ as in Q2}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(c) Let $\vec{y} = F\vec{c} = (0,1,1)^T$, same $\vec{c}$ as above. Is the following equation true? YES or NO. No need to justify, just circle your answer.

$$\begin{pmatrix} y_1 y_1 \\ y_2 y_2 \\ y_3 y_3 \end{pmatrix} = F((F^{-1}\vec{y}) \circledast (F^{-1}\vec{y})),$$

This is the convolution rule, it's always true!

You can rewrite it as $y .\!\times y = F((F^{-1}y) \circledast (F^{-1}y))$ or as

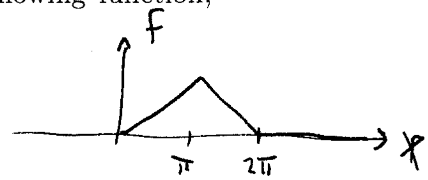
$$(F\vec{c}) .\!\times (F\vec{c}) = F(\vec{c} \circledast \vec{c})$$

You can also check it is true in the special case of $\vec{c}, \vec{y}$ as above:

$$\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ so } \begin{pmatrix} y_1 y_1 \\ y_2 y_2 \\ y_3 y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = F(\vec{c} \circledast \vec{c}) \text{ as calculated in b).}$$

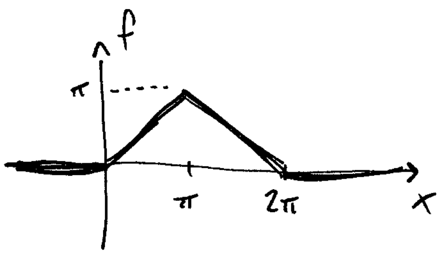
Question 4 ($5+3+6+6+3 = 23$ pts + 5 BONUS) Fourier Integral Transform.

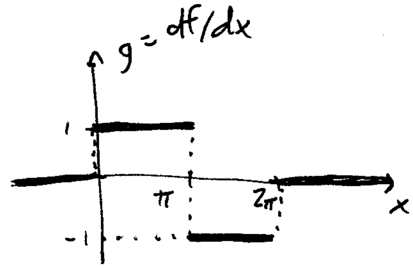
This question uses the Fourier Integral Transform. Let $f(x)$ be the following function, defined over the whole real line $\mathbb{R}$ (NOT 2π-periodic). It looks like a hat.

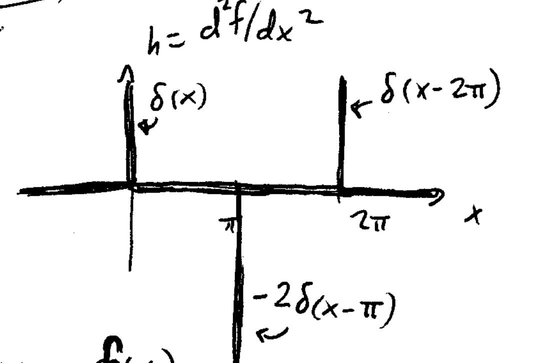


$$f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Draw functions $f(x)$, $g(x) = df/dx$ and $h(x) = d^2f/dx^2$ on 3 separate graphs. Make sure to do this right, it'll be useful in (b), (d) and maybe even (f) ~~(bonus)~~.







(b) The solution $u(x)$ to equation (1) (no calculation needed) is $u(x) =$ $-f(x)$.

(you can see from the graph of h that $\frac{d^2f}{dx^2} = \delta(x) - 2\delta(x-\pi) + \delta(x-2\pi)$)

(1) $$-\frac{d^2u}{dx^2} = \delta(x) - 2\delta(x-\pi) + \delta(x-2\pi)$$

(c) Take the Fourier Integral Transform (FIT) of equation (1) to find an expression for the FIT of $u(x)$, $\hat{u}$. The answer $\hat{u}$ involves only numbers and k's, which means you'll need to calculate the FIT of those delta functions.

$$-\frac{d^2u}{dx^2} = \delta(x) - 2\delta(x-\pi) + \delta(x-2\pi)$$

$$\Big\downarrow \text{FIT}$$

$$-(ik)^2\hat{u} = 1 - 2\cdot 1\cdot e^{-ik\pi} + 1\cdot e^{-ik2\pi} \qquad (\star)$$

shift! shift!

$$k^2\hat{u} = 1 - 2e^{-ik\pi} + e^{-ik2\pi}$$

$$\boxed{\hat{u}(k) = \frac{1 - 2e^{-ik\pi} + e^{-ik2\pi}}{k^2}}$$

Note: because k now is ANY number, not just an integer, we can't say $e^{-ik\pi} = (-1)^k$ anymore!

Question 4 continued. (5 + 3 + 6 + 6 + 3 = 23 **pts + 5 BONUS**)

(d) Find the FIT $\hat{g}(k)$ of $g(x) = df/dx$ using the formula $\hat{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx$.

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x) e^{-ikx}\,dx = \int_0^{\pi} e^{-ikx}\,dx + \int_{\pi}^{2\pi} (-1) e^{-ikx}\,dx$$

$$= \frac{e^{-ikx}}{-ik}\Big|_0^{\pi} - \frac{e^{-ikx}}{-ik}\Big|_{\pi}^{2\pi} = \frac{e^{-ik\pi} - 1 - e^{-ik2\pi} + e^{-ik\pi}}{-ik}$$

$$\boxed{\hat{g}(k) = \frac{-1 + 2e^{-ik\pi} - e^{-ik2\pi}}{-ik}}$$

(again, can't simplify $e^{-ik\pi}$ to $(-1)^k$)

(e) From your answer in (d), find the FIT $\hat{f}(k)$ of $f(x)$.

From the differentiation/integration rule: $\hat{f}(k) \cdot ik = \hat{g}(k)$ if $g(x) = \frac{d}{dx} f(x)$

so that $\hat{f}(k) = \dfrac{\hat{g}(k)}{ik} = \boxed{\dfrac{-1 + 2e^{-ik\pi} - e^{-ik2\pi}}{k^2} = \hat{f}(k)}$

(Verification: since $u(x) = -f(x)$, then $\hat{u}(k) = -\hat{f}(k)$, which is the case

from (b) from (c))

You can check your answer using (b) and (c) if you want! Only partial credit awarded if you use (b) and (c) but not (d) to find $\hat{f}(k)$.

(f) BONUS QUESTION. OPTIONAL. Find the FIT $\hat{h}(k)$ of $h(x)$. You may use either the work you did in (c) or in (d) (or use both to check your answer if you'd like).

$h(x) = \delta(x) - 2\delta(x-\pi) + \delta(x - 2\pi)$, and we can take the FIT of this equation, using equation $(\bigstar)$ in (c), to get

$$\boxed{\hat{h}(k) = 1 - 2e^{-ik\pi} + e^{-ik2\pi}}$$

<u>OR</u>: $h(x) = \frac{dg}{dx}(x)$ so $\hat{h}(k) = ik\,\hat{g}(k)$, which gives the same answer.

Question 5 $(6+6+5+3=20$ **pts) Filtering.**

Consider the following filter, which takes an infinite vector input $\vec{x}$ and outputs the infinite vector $\vec{y}$ as such:

$$y_k = x_k - \frac{1}{2}x_{k-1} - \frac{1}{2}x_{k-2}, \qquad -\infty \le k \le \infty, \qquad k \text{ integer.}$$

(a) Let $\vec{h}$ be the infinite vector such that the convolution $\vec{y} = \vec{h} * \vec{x}$ holds. Write down the components of $\vec{h}$.

the convolution says $y_k = \sum_{l=-\infty}^{\infty} h_{k+l}\, x_{-l} = \underbrace{x_k - \frac{1}{2}x_{k-1} - \frac{1}{2}x_{k-2}}_{\text{filter}}$

so clearly when $-l = k$, $h_{k+l} = h_{k-k} = h_0 = 1$

when $-l = k-1$, $h_{k+l} = h_{k-k+1} = h_1 = -1/2$

$-l = k-2$, $h_{k+l} = h_{k-k+2} = h_2 = -1/2$

all other h_k's are 0!

(b) Let the input have a frequency of ω: $x_k = e^{ik\omega}$. Calculate the output y_k. Also, write the output as $y_k = x_k H(\omega)$. What is this function $H(\omega)$?

$$y_k = e^{ik\omega} - \frac{1}{2}e^{i(k-1)\omega} - \frac{1}{2}e^{i(k-2)\omega} = e^{ik\omega}\left(1 - \frac{1}{2}e^{-i\omega} - \frac{1}{2}e^{-2i\omega}\right)$$

$$\boxed{H(\omega) = 1 - \frac{1}{2}e^{-i\omega} - \frac{1}{2}e^{-2i\omega}}$$

(c) Fill the blanks: $H(0) =$ ___0___ and $H(\pi) =$ ___1___. (If you couldn't do (b), you might still be able to try this one out.)

$$H(0) = 1 - \frac{1}{2}e^0 - \frac{1}{2}e^0 = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

$$H(\pi) = 1 - \frac{1}{2}e^{-i\pi} - \frac{1}{2}e^{-i2\pi} = 1 - \frac{1}{2}(-1) - \frac{1}{2}(1) = 1$$

(d) Is this filter LOWPASS or HIGHPASS (circle one)? No justification needed.

(because $H(\pi) = 1$ ← passes high frequency but $H(0) = 0$ ← does not pass low frequency)