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18.085: Homework 9

Section 4.3 (Problem 16):

If the equation $\det(P-\lambda I)=0$ reduces to $\lambda^4=1$, then the eigenvalues of P are $\pm 1, \pm i$. The permutation matrix which has eigenvalues = cube roots of 1 is

$$P=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

Section 4.3 (Problem 17):

Note that

$$\begin{bmatrix}c_0&c_1&c_2&c_3\\c_3&c_0&c_1&c_2\\c_2&c_3&c_0&c_1\\c_1&c_2&c_3&c_4\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=(c_0+c_1+c_2+c_3)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad \Rightarrow \quad \lambda_0=c_0+c_1+c_2+c_3$$

$$\begin{bmatrix}c_0&c_1&c_2&c_3\\c_3&c_0&c_1&c_2\\c_2&c_3&c_0&c_1\\c_1&c_2&c_3&c_0\end{bmatrix}\begin{bmatrix}1\\i\\i^2\\i^3\end{bmatrix}=\begin{bmatrix}c_0+ic_1-c_2-ic_3\\ic_0-c_1-ic_2+c_3\\-c_0-ic_1+c_2+ic_3\\-ic_0+c_1+ic_2-c_3\end{bmatrix}=(c_0+ic_1-c_2-ic_3)\begin{bmatrix}1\\i\\i^2\\i^3\end{bmatrix} \quad \Rightarrow \quad \lambda_1=c_0+ic_1+i^2c_2+i^3c_3$$

Since $=c_0I+c_1P+c_2P^2+c_3P^3$, where P is the cyclic permutation, then $C=F(c_0I+c_1\Lambda+c_2\Lambda^2+c_3\Lambda^3)F^{-1}$ and the matrix in parenthesis is diagonal and contains the eigenvalues of C.

Section 4.3 (Problem 18):

If C is the matrix shown below

$$C=\begin{bmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{bmatrix} \quad \Rightarrow \quad 2I-\Lambda-\Lambda^3=\begin{bmatrix}0&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}$$

then its eigenvalues are the diagonal entries of $2I-\Lambda-\Lambda^3$ which are 0,2,2 and 4.

Section 4.3 (Problem 20):

If $C=FEF^{-1}$ and we'd like to perform the multiplication Cx as $F\big(E(F^{-1}x)\big)$, then $\frac{n}{2}\log_2 n$ multiplications will come from F^{-1}, $\frac{n}{2}\log_2 n$ multiplications will come from F and n multiplications will come from E.

Section 4.4 (Problem 2):

Note that $F(c\otimes d)=F(c).*F(d)$ as

$$F(c\otimes d)=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}c_0d_0+c_1d_1\\c_0d_1+c_1d_0\end{bmatrix}=\begin{bmatrix}c_0d_0+c_1d_1+c_0d_1+c_1d_0\\c_0d_0+c_1d_1-c_0d_1-c_1d_0\end{bmatrix}=\begin{bmatrix}(c_0+c_1)(d_0+d_1)\\(c_0-c_1)(d_0-d_1)\end{bmatrix}$$

$$F(c).*F(d)=\begin{bmatrix}c_0+c_1\\c_0-c_1\end{bmatrix}.*\begin{bmatrix}d_0+d_1\\d_0-d_1\end{bmatrix}=\begin{bmatrix}(c_0+c_1)(d_0+d_1)\\(c_0-c_1)(d_0-d_1)\end{bmatrix}$$

Section 4.4 (Problem 8):

(a) If $f = (0,0,0,1,0,0)$ is represented by w^3 then $f \otimes f = (1,0,0,0,0,0)$ as it represents $w^3 \cdot w^3 = w^6 = 1$.

(b) $c = F^{-1}f = \frac{1}{6}(1, w^{-3}, w^{-6}, w^{-9}, w^{-12}, w^{-15})^T = \frac{1}{6}(1,-1,1,-1,1,-1)^T$

(c) To obtain $f \otimes f$, note that

$$f \otimes f = 6F(c.*c) = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathrm{w} & \mathrm{w}^2 & \mathrm{w}^3 & \mathrm{w}^4 & \mathrm{w}^5 \\ 1 & \mathrm{w}^2 & \mathrm{w}^4 & \mathrm{w}^6 & \mathrm{w}^8 & \mathrm{w}^{10} \\ 1 & \mathrm{w}^3 & \mathrm{w}^6 & \mathrm{w}^9 & \mathrm{w}^{12} & \mathrm{w}^{15} \\ 1 & \mathrm{w}^4 & \mathrm{w}^8 & \mathrm{w}^{12} & \mathrm{w}^{16} & \mathrm{w}^{20} \\ 1 & \mathrm{w}^5 & \mathrm{w}^{10} & \mathrm{w}^{15} & \mathrm{w}^{20} & \mathrm{w}^{25} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Section 4.4 (Problem 9):

From the entries of $CC^T = C^TC$ which are shown below, we can see that $(1,3,4) * (4,3,1) = (4,15,26,15,4)$ and $(1,2,4) * (4,2,1) = (4,15,26,15,4)$

$$C = \begin{bmatrix} \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ 4 & 3 & 1 & 0 & 0 \\ \cdot & 4 & 3 & 1 & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot \end{bmatrix} \Rightarrow CC^T = \begin{bmatrix} \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ 4 & 3 & 1 & 0 & 0 \\ \cdot & 4 & 3 & 1 & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & \cdot & 4 & \cdot & \cdot \\ \cdot & 1 & 3 & 4 & \cdot \\ 0 & 0 & 1 & 3 & 4 \\ \cdot & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 26 & 15 & 4 & 0 & 0 & \cdot \\ \cdot & 15 & 26 & 15 & 4 & 0 & \cdot \\ \cdot & 4 & 15 & 26 & 15 & 4 & \cdot \\ \cdot & 0 & 4 & 15 & 26 & 15 & \cdot \\ \cdot & 0 & 0 & 4 & 15 & 26 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$C^TC = \begin{bmatrix} \cdot & \cdot & 4 & \cdot & \cdot \\ \cdot & 1 & 3 & 4 & \cdot \\ 0 & 0 & 1 & 3 & 4 \\ \cdot & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ 4 & 3 & 1 & 0 & 0 \\ \cdot & 4 & 3 & 1 & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 26 & 15 & 4 & 0 & 0 & \cdot \\ \cdot & 15 & 26 & 15 & 4 & 0 & \cdot \\ \cdot & 4 & 15 & 26 & 15 & 4 & \cdot \\ \cdot & 0 & 4 & 15 & 26 & 15 & \cdot \\ \cdot & 0 & 0 & 4 & 15 & 26 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$C = \begin{bmatrix} \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ 4 & 2 & 1 & 0 & 0 \\ \cdot & 4 & 2 & 1 & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot \end{bmatrix} \Rightarrow CC^T = \begin{bmatrix} \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ 4 & 2 & 1 & 0 & 0 \\ \cdot & 4 & 2 & 1 & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & \cdot & 4 & \cdot & \cdot \\ \cdot & 1 & 2 & 4 & \cdot \\ 0 & 0 & 1 & 2 & 4 \\ \cdot & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 21 & 10 & 4 & 0 & 0 & \cdot \\ \cdot & 10 & 21 & 10 & 4 & 0 & \cdot \\ \cdot & 4 & 10 & 21 & 10 & 4 & \cdot \\ \cdot & 0 & 4 & 10 & 21 & 10 & \cdot \\ \cdot & 0 & 0 & 4 & 10 & 21 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$C^TC = \begin{bmatrix} \cdot & \cdot & 4 & \cdot & \cdot \\ \cdot & 1 & 2 & 4 & \cdot \\ 0 & 0 & 1 & 2 & 4 \\ \cdot & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ 4 & 2 & 1 & 0 & 0 \\ \cdot & 4 & 2 & 1 & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 21 & 10 & 4 & 0 & 0 & \cdot \\ \cdot & 10 & 21 & 10 & 4 & 0 & \cdot \\ \cdot & 4 & 10 & 21 & 10 & 4 & \cdot \\ \cdot & 0 & 4 & 10 & 21 & 10 & \cdot \\ \cdot & 0 & 0 & 4 & 10 & 21 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Section 4.5 (Problem 1):

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx = \int_{-\infty}^{0} -e^{ax}e^{-ikx}dx + \int_{0}^{\infty} e^{-ax}e^{-ikx}dx = \int_{-\infty}^{0} -e^{(a-ik)x}dx + \int_{0}^{\infty} e^{-(a+ik)x}dx$$
$$= \frac{-1}{a-ik} + \frac{1}{a+ik} = \frac{-2ik}{a^2+k^2}$$

The decay rate of $\hat{g}(k)$ is $1/k$ and there is a jump (discontinuity) in $g(x)$.

Section 4.5 (Problem 2):

$$(a)\qquad \hat{f}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx=\int_0^L e^{-ikx}dx=\frac{e^{-ikL}-1}{-ik}=\frac{1-e^{-ikL}}{ik}$$

$$(b)\qquad \hat{f}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx=\int_{-\infty}^{0} -e^{-ikx}dx+\int_0^{\infty} e^{-ikx}dx=\frac{2}{ik}=\frac{-2i}{k}$$

$$(c)\qquad f(x)=\int_0^1 e^{ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)\,e^{ikx}dk \quad \text{where } \hat{f}(k)=\begin{cases}2\pi & 0\le k\le 1\\ 0 & \text{otherwise}\end{cases}$$

$$(d)\qquad \hat{f}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx=\int_0^{4\pi}\sin x\, e^{-ikx}dx=\frac{e^{-4i\pi k}-1}{k^2-1}$$

Section 4.5 (Problem 10):

Since $f'(x)=-ae^{-ax}+\delta(x)$, then the transform of $f'(x)$ is not just $-a\hat{f}(k)$ but $ik\hat{f}(k)=\frac{ik}{a+ik}$

Section 4.5 (Problem 11):

For $du/dx+au=\delta(x-d)$, we have $ik\hat{u}(k)+a\hat{u}(k)=e^{-ikd}$ and

$$\hat{u}(k)=\frac{e^{-ikd}}{a+ik}\quad\Rightarrow\quad u(x)=\begin{cases}e^{-a(x-d)} & d\le x\\ 0 & \text{otherwise}\end{cases}$$

Section 4.5 (Problem 12):

For $\int u(x)\,dx-u'(x)=\delta(x)$, we have $\hat{u}(k)/ik-ik\hat{u}(k)=1$ and

$$\hat{u}(k)=\frac{ik}{1+k^2}\quad\Rightarrow\quad u(x)=\begin{cases}-e^{-x}/2 & 0<x\\ e^{x}/2 & 0>x\end{cases}$$

Section 4.5 (Problem 14):

Since $S(x)=\text{Hat}*\text{Hat}=(\text{Box}*\text{Box})*(\text{Box}*\text{Box})$, then

$$\hat{S}(k)=\frac{4(1-\cos k)^2}{k^4}=\frac{4(1+\cos^2 k-2\cos k)}{k^4}=\frac{4\left(1+\frac{1}{4}\left(e^{ki}+e^{-ki}\right)^2-e^{ik}-e^{-ik}\right)}{k^4}$$
$$=\frac{4\left(\frac{3}{2}+\frac{1}{4}e^{2ki}+\frac{1}{4}e^{-2ki}-e^{ik}-e^{-ik}\right)}{k^4}$$

Since $(ik)^4\hat{S}(k)$ is the transform of $S''''(x)$, then $S''''(x)=6\delta(x)+\delta(x+2)+\delta(x-2)-4\delta(x+1)-4\delta(x-1)$ which has spikes at $x=-2,-1,0,1,2$.

Section 4.5 (Problem 21):

Since $g(x)*1=\left(\int\limits_{-\infty}^{\infty} g(t)dt\right)(1)$ then $\int\limits_{-\infty}^{\infty} g(t)dt$ is the eigenvalue associated with the eigenfunction $u(x)=1$

Section 4.5 (Problem 23):

$$\delta(x)*\delta(x)=\delta(x)$$