18.085: Homework 8

## Section 4.1 (Problem 1 parts b and c):

(b) If $f(x)=|\sin x|$, then $f(x)=\sum_{k=0}^{\infty} a_{k} \cos k x$, then

$$
\begin{gathered}
a_{0}=\frac{1}{2 \pi} \int_{-\pi}^{\pi}|\sin x| d x=\frac{1}{\pi} \int_{0}^{\pi} \sin x d x=\frac{2}{\pi} \\
a_{k \geq 1}=\frac{2}{\pi} \int_{0}^{\pi}|\sin x| \cos k x d x=\frac{2}{\pi} \int_{0}^{\pi} \sin x \cos k x d x=\left.\frac{2}{\pi} \frac{k \sin x \sin k x+\cos x \cos k x}{k^{2}-1}\right|_{0} ^{\pi}=\frac{2(\cos k \pi+1)}{\pi\left(1-k^{2}\right)} \\
\quad=\frac{2\left[(-1)^{k}+1\right]}{\pi\left(1-k^{2}\right)}= \begin{cases}0 & k \text { is odd } \\
\frac{4}{\pi\left(1-k^{2}\right)} & k \text { is even }\end{cases}
\end{gathered}
$$

Hence

$$
|\sin x|=\frac{2}{\pi}-\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2 k x}{(2 k)^{2}-1}
$$

(c) If $f(x)=x$, then $f(x)=\sum_{k=1}^{\infty} b_{k} \sin k x$, then

$$
b_{k \geq 1}=\frac{2}{\pi} \int_{0}^{\pi} x \sin k x d x=\left.\frac{2}{\pi} \frac{\sin k x-k x \cos k x}{k^{2}}\right|_{0} ^{\pi}=\frac{-2 \cos \pi k}{k}=\frac{2(-1)^{k+1}}{k}
$$

Hence,

$$
x=2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sin k x}{k}
$$

## Section 4.1 (Problem 3):

For the following even square pulse $f(x)=\sum_{k=0}^{\infty} a_{k} \cos k x$ (all $b^{\prime} s$ are zero as $f$ is even).


$$
\begin{aligned}
a_{0} & =\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x) d x=\frac{1}{2 \pi} \int_{-\pi / 2}^{\pi / 2} 1 d x=\frac{1}{2} \\
a_{k \geq 1} & =\frac{2}{\pi} \int_{0}^{\pi} f(x) \cos k x d x=\frac{2}{\pi} \int_{0}^{\pi / 2}(1) \cos k x d x=\frac{2}{\pi k} \sin \left(\frac{\pi k}{2}\right)=\operatorname{sinc}\left(\frac{\pi k}{2}\right)
\end{aligned}
$$

Hence,

$$
f(x)=\frac{1}{2}+\sum_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \cos k x
$$

## Section 4.1 (Problem 6):

The constant function closest to $f(x)=\cos ^{2} x$ is

$$
a_{0}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x) d x=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \cos ^{2} x d x=\frac{1}{2}
$$

The multiple of $\cos x$ closest to $g(x)=\cos ^{3} x$ is

$$
a_{1}=\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos x d x=\frac{1}{\pi} \int_{-\pi}^{\pi} \cos ^{4} x d x=\frac{3}{4}
$$

## Section 4.1 (Problem 13):

For the following even square pulse $f(x)=\frac{1}{2}+\sum_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \cos k x$

(a) The energy $\int|f(x)|^{2} d x$ is

$$
\int_{-\pi}^{\pi}|f(x)|^{2} d x=\int_{-\pi / 2}^{\pi / 2} 1 d x=\pi
$$

(b) Note that

$$
f(x)=\frac{1}{2}+\sum_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \cos k x=\frac{1}{2}+\sum_{k=1}^{\infty} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \frac{e^{i k x}+e^{-i k x}}{2} \Rightarrow c_{0}=\frac{1}{2} \text { and } c_{k \neq 0}=\frac{1}{2} \operatorname{sinc}\left(\frac{\pi k}{2}\right)
$$

(c) Using the energy identity, and noting that $c_{k}$ is even, we get

$$
\begin{aligned}
\int_{-\pi}^{\pi}|f(x)|^{2} d x= & 2 \pi \sum_{k=-\infty}^{\infty}\left|c_{k}\right|^{2}=2 \pi\left(\left|c_{0}\right|^{2}+2 \sum_{k=1}^{\infty}\left|c_{k}\right|^{2}\right)=2 \pi\left(\frac{1}{4}+\frac{1}{2} \sum_{k=1}^{\infty}\left|\operatorname{sinc}\left(\frac{\pi k}{2}\right)\right|^{2}\right) \\
& =2 \pi\left(\frac{1}{4}+\frac{2}{\pi^{2}} \sum_{k=1}^{\infty} \frac{1}{(2 k-1)^{2}}\right)=2 \pi\left(\frac{1}{4}+\frac{2}{\pi^{2}}\left(\frac{\pi^{2}}{8}\right)\right)=\pi
\end{aligned}
$$

## Section 4.2 (Problem 1):

(a) For the quarter square function $F(x, y)= \begin{cases}1 & \text { for } 0 \leq x \leq \pi, 0 \leq y \leq \pi \\ 0 & \text { if }-\pi<x<0 \text { or }-\pi<y<0\end{cases}$

$$
\begin{aligned}
& c_{m n(m, n \neq 0)}=\frac{1}{4 \pi^{2}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(x, y) e^{-i m x} e^{-i n y} d x d y=\frac{1}{4 \pi^{2}} \int_{0}^{\pi} \int_{0}^{\pi} e^{-i m x} e^{-i n y} d x d y=\frac{1}{4 \pi^{2}} \frac{\left(e^{-i m \pi}-1\right)}{-i m} \frac{\left(e^{-i n \pi}-1\right)}{-i n} \\
& =\frac{-1}{4 \pi^{2}} \frac{\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]}{m n}=\left\{\begin{array}{cl}
0 & \text { if } \mathrm{m} \text { or } \mathrm{n} \text { is even } \\
\frac{-1}{\pi^{2} m n} & \text { if } \mathrm{m} \text { and } \mathrm{n} \text { are odd }
\end{array}\right. \\
& c_{0 n(n \neq 0)}=\frac{1}{4 \pi^{2}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(x, y) e^{-i n y} d x d y=\frac{1}{4 \pi^{2}} \int_{0}^{\pi} \int_{0}^{\pi} e^{-i n y} d x d y=\frac{1}{4 \pi} \frac{\left(e^{-i n \pi}-1\right)}{-i n}=\frac{1-(-1)^{n}}{4 \pi i n} \\
& =\left\{\begin{array}{cl}
0 & \text { if } n \text { is even } \\
\frac{1}{2 \pi i n} & \text { if } n \text { is odd }
\end{array}\right. \\
& c_{m 0(m \neq 0)}=\frac{1}{4 \pi^{2}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(x, y) e^{-i m x} d x d y=\frac{1}{4 \pi^{2}} \int_{0}^{\pi} \int_{0}^{\pi} e^{-i m x} d x d y=\frac{1}{4 \pi} \frac{\left(e^{-i m \pi}-1\right)}{-i m}=\frac{1-(-1)^{m}}{4 \pi i m} \\
& =\left\{\begin{array}{cl}
0 & \text { if } m \text { is even } \\
\frac{1}{2 \pi i m} & \text { if } m \text { is odd }
\end{array}\right.
\end{aligned}
$$

(b) For the checkerboard function $F(x, y)= \begin{cases}1 & \text { if } x y \geq 0-\pi<x \leq \pi \\ 0 & \text { if } x y<0-\pi<y \leq \pi\end{cases}$


$$
\begin{aligned}
c_{m n(m, n \neq 0)}= & \frac{1}{4 \pi^{2}} \int_{0}^{\pi} \int_{0}^{\pi} e^{-i m x} e^{-i n y} d x d y+\frac{1}{4 \pi^{2}} \int_{-\pi}^{0} \int_{-\pi}^{0} e^{-i m x} e^{-i n y} d x d y \\
& =\frac{1}{4 \pi^{2}} \frac{\left(e^{-i m \pi}-1\right)}{-i m} \frac{\left(e^{-i n \pi}-1\right)}{-i n}+\frac{1}{4 \pi^{2}} \frac{\left(1-e^{i m \pi}\right)}{-i m} \frac{\left(1-e^{i n \pi}\right)}{-i n} \\
& =\frac{-1}{4 \pi^{2}} \frac{\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]}{m n}+\frac{-1}{4 \pi^{2}} \frac{\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]}{m n} \\
& =\frac{-1}{2 \pi^{2}} \frac{\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]}{m n}= \begin{cases}0 & \text { if } \mathrm{m} \text { or } \mathrm{n} \text { is even } \\
\frac{-2}{\pi^{2} m n} \text { if } \mathrm{m} \text { and } \mathrm{n} \text { are odd }\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
c_{0 n(n \neq 0)} & =\frac{1}{4 \pi^{2}} \int_{0}^{\pi} \int_{0}^{\pi} e^{-i n y} d x d y+\frac{1}{4 \pi^{2}} \int_{-\pi}^{0} \int_{-\pi}^{0} e^{-i n y} d x d y=\frac{1}{4 \pi} \frac{\left(e^{-i n \pi}-1\right)}{-i n}+\frac{1}{4 \pi} \frac{\left(1-e^{i n \pi}\right)}{-i n}=0 \\
c_{m 0(m \neq 0)} & =\frac{1}{4 \pi^{2}} \int_{0}^{\pi} \int_{0}^{\pi} e^{-i m x} d x d y+\frac{1}{4 \pi^{2}} \int_{-\pi}^{0} \int_{-\pi}^{0} e^{-i m x} d x d y=\frac{1}{4 \pi} \frac{\left(e^{-i m \pi}-1\right)}{-i m}+\frac{1}{4 \pi} \frac{\left(1-e^{i m \pi}\right)}{-i m}=0
\end{aligned}
$$

## Section 4.2 (Problem 3):

In the double sine series, that is $f(x, y)=\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m n} \sin (m x) \sin (n y)$, the formula for $b_{m n}$ is

$$
b_{m n}=\frac{1}{\pi^{2}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \sin (m x) \sin (n y) d x d y
$$

## Section 4.3 (Problem 2):

Note that row $k$ of $\bar{F}$ is

$$
\overline{F_{k}}=\left[\begin{array}{llll}
1 & \bar{w}_{N}^{k} & \bar{w}_{N}^{2 k} & \bar{w}_{N}^{3 k} \ldots
\end{array} \bar{w}_{N}^{(N-1) k}\right]
$$

Using $w^{N}=1$ we get $w_{N}^{N-k}=w_{N}^{-k}=e^{-2 \pi k i / N}=\left(\overline{e^{2 \pi l / N}}\right)^{k}=\bar{w}_{N}^{k}$; therefore, row $N-k$ of $F$ is

$$
\begin{gathered}
F_{N-k}=\left[\begin{array}{cccc}
1 w_{N}^{N-k} & w_{N}^{2(N-k)} w_{N}^{3(N-k)} & \ldots & w_{N}^{(N-1)(N-k)}
\end{array}\right]=\left[\begin{array}{llll}
1 w_{N}^{-k} & w_{N}^{-2 k} & w_{N}^{-3 k} \ldots & w_{N}^{-(N-1) k}
\end{array}\right] \\
=\left[\begin{array}{llll}
1 \bar{w}_{N}^{k} & \bar{w}_{N}^{2 k} & \bar{w}_{N}^{3 k} \ldots & \bar{w}_{N}^{(N-1) k}
\end{array}\right]=\overline{F_{k}}
\end{gathered}
$$

## Section 4.3 (Problem 6):

With $w_{6}=e^{2 \pi i / 6}=e^{\pi i / 3}=w_{3}^{1 / 2}$, we have

$$
\begin{aligned}
& F_{6}=\left[\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & w_{6} & 0 \\
0 & 0 & 1 & 0 & 0 & w_{6}{ }^{2} \\
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & -w_{6} & 0 \\
0 & 0 & 1 & 0 & 0 & -w_{6}^{2}
\end{array}\right]\left[\begin{array}{cccccc}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & w_{6}^{2} & w_{6}^{4} & 0 & 0 & 0 \\
1 & w_{6}^{4} & w_{6}^{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & w_{6}^{2} & w_{6}^{4} \\
0 & 0 & 0 & 1 & w_{6}^{4} & w_{6}^{2}
\end{array}\right]\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]= \\
&=\left[\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & w_{6}^{2} & w_{6}^{4} & w_{6} & w_{6}^{3} & w_{6}^{5} \\
1 & w_{6}^{4} & w_{6}^{8} & w_{6}^{2} & w_{6}^{6} & w_{6}^{10} \\
1 & 1 & 1 & -1 & -1 & -1 \\
1 & w_{6}^{2} & w_{6}^{4} & -w_{6} & -w_{6}^{3} & -w_{6}^{5} \\
1 & w_{6}^{4} & w_{6}^{8} & -w_{6}^{2} & -w_{6}^{6} & -w_{6}^{10}
\end{array}\right]\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1
\end{array}\right]= \\
&=\left[\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & w_{6} & w_{6}^{2} & w_{6}^{3} & w_{6}^{4} & w_{6}^{5} \\
1 & w_{6}^{2} & w_{6}^{4} & w_{6}^{6} & w_{6}^{8} & w_{6}^{10} \\
1 & -1 & 1 & -1 & 1 & -1 \\
1 & -w_{6} & w_{6}^{2} & -w_{6}^{3} & w_{6}^{4} & -w_{6}^{5} \\
1 & -w_{6}^{2} & w_{6}^{4} & -w_{6}^{6} & w_{6}^{8} & -w_{6}^{10}
\end{array}\right]=\left[\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & w_{6} & w_{6}^{2} & w_{6}^{3} & w_{6}^{4} & w_{6}^{5} \\
1 & w_{6}^{2} & w_{6}^{4} & w_{6}^{6} & w_{6}^{8} & w_{6}^{10} \\
1 & w_{6}^{3} & w_{6}^{6} & w_{6}^{9} & w_{6}^{12} & w_{6}^{15} \\
1 & w_{6}^{4} & w_{6}^{8} & w_{6}^{12} & w_{6}^{16} & w_{6}^{20} \\
1 & w_{6}^{5} & w_{6}^{10} & w_{6}^{15} & w_{6}^{20} & w_{6}^{25}
\end{array}\right]
\end{aligned}
$$

## Section 4.3 (Problem 8):

Split $c=(1,0,1,0)$ into $c_{\text {even }}=\left(c_{0}, c_{2}\right)=(1,1)$ and $c_{\text {odd }}=\left(c_{1}, c_{3}\right)=(0,0)$. Transform them separately by $F_{2}$ into $f_{\text {even }}$ and $f_{\text {odd }}$ where

$$
\begin{gathered}
f_{\text {even }}=F_{2} c_{\text {even }}^{T}=\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right] \quad \text { and } \quad f_{\text {odd }}=F_{2} c_{\text {odd }}^{T}=\left[\begin{array}{ll}
1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
0 \\
0
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \\
f_{1 \text { st half }}=f_{\text {even }}+\left[\begin{array}{cc}
w_{4}^{0} & 0 \\
0 & w_{4}^{1}
\end{array}\right] f_{\text {odd }}=\left[\begin{array}{l}
2 \\
0
\end{array}\right]+\left[\begin{array}{ll}
1 & 0 \\
0 & i
\end{array}\right]\left[\begin{array}{l}
0 \\
0
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right] \\
f_{\text {2nd half }}=f_{\text {even }}-\left[\begin{array}{cc}
w_{4}^{0} & 0 \\
0 & w_{4}^{1}
\end{array}\right] f_{\text {odd }}=\left[\begin{array}{l}
2 \\
0
\end{array}\right]-\left[\begin{array}{ll}
1 & 0 \\
0 & i
\end{array}\right]\left[\begin{array}{l}
0 \\
0
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right]
\end{gathered}
$$

Combine to get $f=\left[\begin{array}{lll}2 & 0 & 2\end{array} 0^{T}\right.$

Split $c=(0,1,0,1)$ into $c_{\text {even }}=\left(c_{0}, c_{2}\right)=(0,0)$ and $c_{\text {odd }}=\left(c_{1}, c_{3}\right)=(1,1)$. Transform them separately by $F_{2}$ into $f_{\text {even }}$ and $f_{\text {odd }}$ where

$$
\begin{gathered}
f_{\text {even }}=F_{2} c_{\text {even }}^{T}=\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
0 \\
0
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \quad \text { and } \quad f_{\text {odd }}=F_{2} c_{\text {odd }}^{T}=\left[\begin{array}{ll}
1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right] \\
f_{1 \text { st half }}=f_{\text {even }}+\left[\begin{array}{cc}
w_{4}^{0} & 0 \\
0 & w_{4}^{1}
\end{array}\right] f_{\text {odd }}=\left[\begin{array}{l}
0 \\
0
\end{array}\right]+\left[\begin{array}{ll}
1 & 0 \\
0 & i
\end{array}\right]\left[\begin{array}{l}
2 \\
0
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right] \\
f_{2 \text { nd half }}=f_{\text {even }}-\left[\begin{array}{cc}
w_{4}^{0} & 0 \\
0 & w_{4}^{1}
\end{array}\right] f_{\text {odd }}=\left[\begin{array}{l}
0 \\
0
\end{array}\right]-\left[\begin{array}{ll}
1 & 0 \\
0 & i
\end{array}\right]\left[\begin{array}{l}
2 \\
0
\end{array}\right]=\left[\begin{array}{c}
-2 \\
0
\end{array}\right]
\end{gathered}
$$

Combine to get $f=\left[\begin{array}{lll}2 & 0 & -2\end{array}\right]^{T}$

## Section 4.3 (Problem 8):

(a) The 6 roots of $w^{6}=1$ are shown below, note that
$w+w^{2}+w^{3}+w^{4}+w^{5}+w^{6}=\left(w+w^{4}\right)+\left(w^{2}+w^{5}\right)+\left(w^{3}+w^{6}\right)=(w-w)+\left(w^{2}-w^{2}\right)+\left(w^{3}-w^{3}\right)=0$


$$
w+w^{2}+w^{3}+w^{4}+w^{5}+w^{6}=1+w+w^{2}+w^{3}+w^{4}+w^{5}=\frac{w^{6}-1}{w-1}=\frac{1-1}{w-1}=0
$$

(b) The three cube roots of 1 are shown below and sum to zero.

$$
w_{6}^{2}=-\frac{1}{2}+\frac{\sqrt{3}}{2} i, w_{6}^{4}=-\frac{1}{2}-\frac{\sqrt{3}}{2} i, w_{6}^{6}=1 \quad \Rightarrow \quad w_{6}^{2}+w_{6}^{4}+w_{6}^{6}=0
$$

## MATLAB problem:

The Fourier sine series of $f(x)=x$ is $x=2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sin k x}{k}$. The graph of the Fourier series of $f(x)=x$ up to $\sin 10 x$ and up to $\sin 20 x$ are shown below

$(a+b)$ The maximum error is the absolute value of the first local minima or maxima near $-\pi$ and $\pi$ respectively. If we plot these errors and magnify the output at the local max/min, we get the following results

$$
\begin{gathered}
\operatorname{error}_{10}=x-2 \sum_{k=1}^{10} \frac{(-1)^{k+1} \sin k x}{k} \quad \text { maximum error }=0.5644 \\
d_{10}=\text { Maximum error is at a distance } \approx 0.30189 \text { from } \pi
\end{gathered}
$$



$$
\begin{gathered}
\operatorname{error}_{20}=x-2 \sum_{k=1}^{20} \frac{(-1)^{k+1} \sin k x}{k} \quad \text { maximum error }=0.5628 \\
d_{20}=\text { Maximum error is at a distance } \approx 0.153993 \text { from } \pi
\end{gathered}
$$



From these results, we can see that $d_{20} \approx \frac{1}{2} d_{10}$
(c) First note that

$$
x=\sum_{k=1}^{\infty} 2 \frac{(-1)^{k+1} \sin k x}{k}=\sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k}\left(\frac{e^{i k x}-e^{-i k x}}{2 i}\right)=\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k i} e^{i k x}+\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k i} e^{-i k x}
$$

The energy $\int|f(x)|^{2} d x$ is

$$
\int_{-\pi}^{\pi}|f(x)|^{2} d x=\int_{-\pi / 2}^{\pi / 2} x^{2} d x=\frac{2 \pi^{3}}{3}
$$

Now we have

$$
2 \pi \sum_{k=-\infty}^{\infty}\left|c_{k}\right|^{2}=2 \pi\left(2 \sum_{k=1}^{\infty} \frac{1}{k^{2}}\right)=2 \pi\left(2\left(\frac{\pi^{2}}{6}\right)\right)=\frac{2 \pi^{3}}{3}=\int_{-\pi}^{\pi}|f(x)|^{2} d x
$$

When $k=10$ and $k=20$, we have

$$
\begin{aligned}
& 2 \pi \sum_{k=-10}^{10}\left|c_{k}\right|^{2} \approx 0.942146 \int_{-\pi}^{\pi}|f(x)|^{2} d x \approx 19.475 \\
& 2 \pi \sum_{k=-20}^{20}\left|c_{k}\right|^{2} \approx 0.970351 \int_{-\pi}^{\pi}|f(x)|^{2} d x \approx 20.058
\end{aligned}
$$

(d) The error at $x=\pi / 2$ is $e_{10}=0.099$ and $e_{20}=0.0498$ when we truncate Fourier series with 10 and 20 terms respectively. These numbers are shown below.



When we truncate Fourier series with 30 terms, we expect the error at $x=\pi / 2$ to be $e_{30} \approx \frac{1}{3} e_{10}=0.0333$ (Decay rate is $1 / k$ ) which is identical to the true result shown below


## MATLAB code:

function $F=$ Fourier (M)
$\mathrm{M}=30$;
\% Part zero
$\mathrm{x}=1$ inspace (-pi,pi, 100*M) ;
sum1 $=0$; sum2 $=0$; sum3=0;
for $k=1: M$
if $k<=10$
$\operatorname{sum} 1=\operatorname{sum} 1+2 *\left((-1)^{\wedge}(k+1)\right) * \sin (k * x) / k$; end
if $k<=20$
sum2 $=\operatorname{sum} 2+2^{*}\left((-1)^{\wedge}(k+1)\right) * \sin (k * x) / k$; end sum3 $=\operatorname{sum} 3+2 *\left((-1)^{\wedge}(k+1)\right) * \sin (k * x) / k$;
end
sfigure (1)
splot ( x, sum1, x, sum2, 'LineWidth', 1.5); grid on;
\%xlabel('x'); ylabel('f(x)');
stitle('Graph of the fourier series of $f(x)=x$ ')
\%Part (a+b)
diff=x-sum3;
figure (2)
plot(x,diff,'LineWidth', 1.5); grid on
xlabel('x'); ylabel('f(x)');
title('Plot of the Error of $f(x)$ - truncated series')
\%Part (d)
$\mathrm{m}=\mathrm{pi} / 2$; sum4=0;
for $k=1: M$ $\operatorname{sum} 4=\operatorname{sum} 4+2^{*}\left((-1)^{\wedge}(k+1)\right) * \sin (k * m) / k ;$
end
ErrorAtPiOver2=abs (m-sum4)

