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18.085: Homework 7

Section 3.3 (Problem 6):

For $v = (y^2, x^2)$, curl $v = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 2x - 2y \neq 0$; therefore, v is not a gradient of any function u.
For $v = (y^2, 2xy)$, curl $v = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 2y - 2y = 0$; therefore, v is a gradient of some u which is $u = xy^2 + C$

Section 3.3 (Problem 7):

For $w = (x^2, y^2)$, div $w = 2x + 2y \neq 0$; therefore, w does not have the form $(\partial s/\partial y, -\partial s/\partial x)$ for any function s.
For $w = (y^2, x^2)$, div $w = 0 + 0 = 0$; therefore, w has the form $(\partial s/\partial y, -\partial s/\partial x)$ for some function s which is $(-x^3 + y^3)/3 + C$

Section 3.3 (Problem 8):

If $u = x^2$ in the square $S = \{-1 < x, y < 1\}$, then $w = \text{grad } u = (2x, 0)$ and

$$\begin{aligned}\int_C n\cdot\text{grad}\,u\,ds &= \int_{\text{Left}} n\cdot\text{grad}\,u\,ds + + \int_{\text{Top}} n\cdot\text{grad}\,u\,ds + \int_{\text{Right}} n\cdot\text{grad}\,u\,ds + \int_{\text{Bottom}} n\cdot\text{grad}\,u\,ds \\ &= \int_{-1}^{1}(-i)\cdot(2xi)\,dy + \int_{-1}^{1}(j)\cdot(2xi)\,dx + \int_{-1}^{1}(i)\cdot(2xi)dy + \int_{-1}^{1}(-j)\cdot(2xi)\,dx \\ &= \int_{-1}^{1}(-1)\big(2(-1)\big)\,dy + 0 + \int_{-1}^{1}(1)\big(2(1)\big)dy + 0 = 2(2) + 2(2) = 8\end{aligned}$$

$$\iint_S \text{div grad}\,u\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} 2dxdy = 2(2)(2) = 8$$

Section 3.3 (Problem 27):

For the following saddle point matrix

$$M = \begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix} = \begin{bmatrix} \text{curl} & \text{grad} \\ -\text{div} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y & \partial/\partial x \\ \partial/\partial z & 0 & -\partial/\partial x & \partial/\partial y \\ -\partial/\partial y & \partial/\partial x & 0 & \partial/\partial z \\ -\partial/\partial x & -\partial/\partial y & -\partial/\partial z & 0 \end{bmatrix}$$

Note that

$$M^2 = \begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y & \partial/\partial x \\ \partial/\partial z & 0 & -\partial/\partial x & \partial/\partial y \\ -\partial/\partial y & \partial/\partial x & 0 & \partial/\partial z \\ -\partial/\partial x & -\partial/\partial y & -\partial/\partial z & 0 \end{bmatrix}\begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y & \partial/\partial x \\ \partial/\partial z & 0 & -\partial/\partial x & \partial/\partial y \\ -\partial/\partial y & \partial/\partial x & 0 & \partial/\partial z \\ -\partial/\partial x & -\partial/\partial y & -\partial/\partial z & 0 \end{bmatrix}$$

$$= -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -\Delta^2 I_4$$

Hence,

$$M^2 = \begin{bmatrix} \text{curl} & \text{grad} \\ -\text{div} & 0 \end{bmatrix}\begin{bmatrix} \text{curl} & \text{grad} \\ -\text{div} & 0 \end{bmatrix} = \begin{bmatrix} \text{curl curl} - \text{grad div} & \text{curl grad} \\ -\text{div curl} & -\text{div grad} \end{bmatrix} = \begin{bmatrix} -\Delta^2 I_2 & 0 \\ 0 & -\Delta^2 I_2 \end{bmatrix}$$

and the following identities curl curl $-$ grad div $= -\Delta^2$, curl grad $= 0$ and div curl $= 0$ are verified.

Section 3.4 (Problem 6):

With $|z| = R$ note that $2X = (R + R^{-1})\cos\theta$ and $2Y = (R - R^{-1})\sin\theta$. By setting $r = R$ in the formula below, we get

$$\frac{4X^2}{(r + r^{-1})^2} + \frac{4Y^2}{(r - r^{-1})^2} = \frac{4X^2}{(R + R^{-1})^2} + \frac{4Y^2}{(R - R^{-1})^2} = 1$$

With $|z| = R^{-1}$ note that $2X = (R^{-1} + R)\cos\theta$ and $2Y = (R^{-1} - R)\sin\theta$. By setting $r = R^{-1}$, in the formula below, we get

$$\frac{4X^2}{(r + r^{-1})^2} + \frac{4Y^2}{(r - r^{-1})^2} = \frac{4X^2}{(R^{-1} + R)^2} + \frac{4Y^2}{(R^{-1} - R)^2} = \frac{4X^2}{(R + R^{-1})^2} + \frac{4Y^2}{(R - R^{-1})^2} = 1$$

Hence, $|z| = r$ and $|z| = r^{-1}$ produce the same ellipse as that indicated in the problem.

Section 3.4 (Problem 18):

If $u_0 = \sum B_k \sin \pi k x$, along the edge $y = 0$ of a square and zero on the other three edges, then

$$u(x,y) = \sum B_k\, u_k(x,y) \qquad \text{where } u_k(x,y) = \frac{(\sin \pi k x)(\sinh \pi k(1-y))}{\sinh \pi k}$$

Section 3.4 (Problem 20):

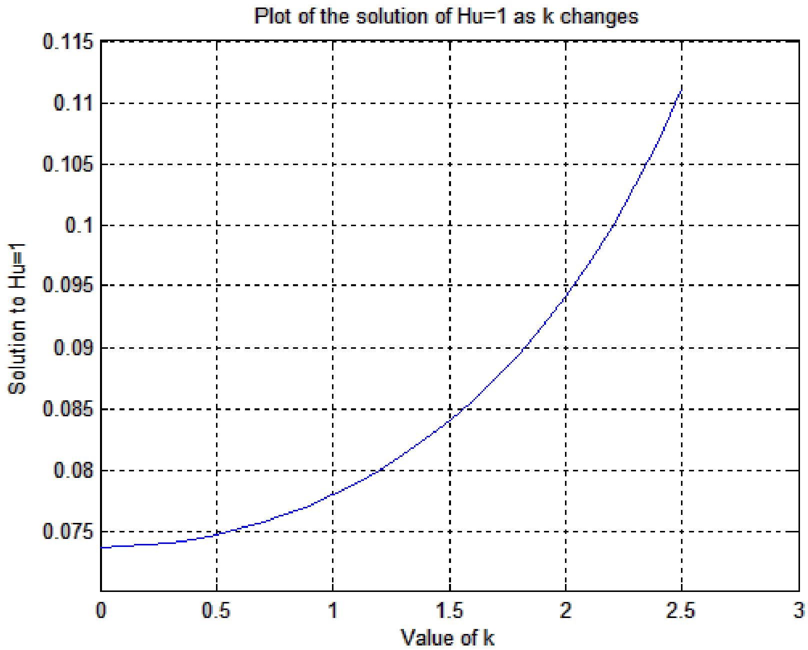
The eigenvalues of $Hu = -u_{xx} - u_{yy} - k^2 u$ are $(m^2 + n^2)\pi^2 - k^2$ as

$$\begin{aligned} H[(\sin \pi k x)(\sin \pi k y)] &= m^2\pi^2(\sin \pi k x)(\sin \pi k y) + n^2\pi^2(\sin \pi k x)(\sin \pi k y) - k^2(\sin \pi k x)(\sin \pi k y) \\ &= [(m^2 + n^2)\pi^2 - k^2](\sin \pi k x)(\sin \pi k y) \end{aligned}$$

The code to solve $Hu = 1$ is shown below and the output of the code is shown in the following page.

```
function P = poisson(x,y)
N=39; u=zeros(size(x)); k=0; M=1;

while k < sqrt(2*pi)
      u=0;
     for i=1:2:N
         for j=1:2:N
             u=u+2^4/pi^2/(i*j)/(pi^2*(i^2+j^2)-k^2)*sin(i*pi*x)*sin(j*pi*y);
         end;
     end;
     W(M)=k; Solution(M)=u; M=M+1; k=k+0.1;
end;

Solution

plot(W,Solution,'-')
xlabel('Value of k') ; grid on;
ylabel('Solution to Hu=1')
title('Plot of the solution of Hu=1 as k changes')
```

```
EDU>> poisson(1/2,1/2)

Solution =

  Columns 1 through 10

    0.0737    0.0737    0.0738    0.0740    0.0743    0.0747    0.0752    0.0757    0.0764    0.0771

  Columns 11 through 20

    0.0780    0.0789    0.0800    0.0812    0.0825    0.0840    0.0856    0.0875    0.0895    0.0917

  Columns 21 through 26

    0.0941    0.0968    0.0998    0.1032    0.1069    0.1111
```

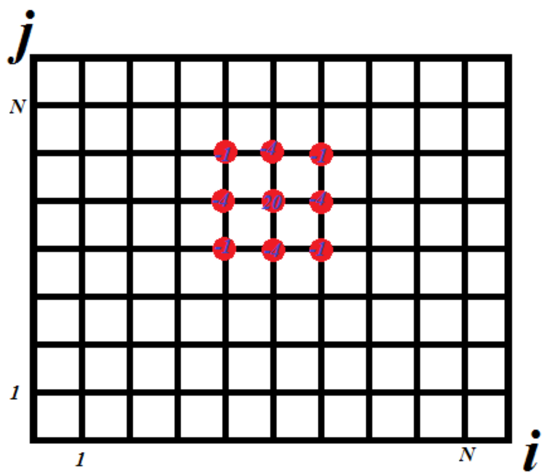


Section 3.5 (Problem 1):

$$\mathrm{K3D} = \mathrm{kron}(\mathrm{K2D}, \mathrm{I}) + \mathrm{kron}(\mathrm{I2D}, \mathrm{K})$$

Section 3.5 (Problem 5):

The 9-points molecule with coefficients 20,-4 and -1 is shown below. The bandwidth of K2D9 is $N + 1$ where N is shown in the figure below (Here we have defined w as the # of total elements starting from the -4 east of 20 to the -1 north east of 20 including those -4 and -1). If we define the bandwidth to be the number of meshes between 20 and -1 north east of 20, then bandwidth is N.



Section 3.6 (Problem 5):

With $h = 1/2$, we have a mesh containing 8 triangles and 9 nodes (8 of them are boundary nodes) which are shown below along with p, t and b

p =

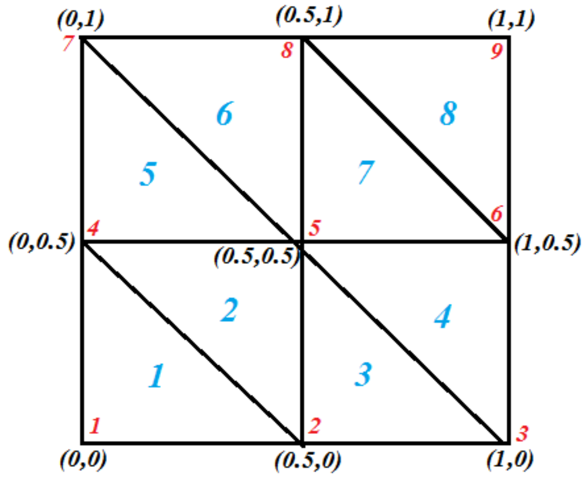
0	0
0.5000	0
1.0000	0
0	0.5000
0.5000	0.5000
1.0000	0.5000
0	1.0000
0.5000	1.0000
1.0000	1.0000

t =

1	2	4
2	4	5
2	3	5
3	5	6
4	5	7
5	7	8
5	8	6
6	8	9

b =

1
2
3
4
6
7
8
9



The 9 by 9 singular matrix K and load vector F (before setting the boundary as 0) for $f(x) = 1$ are shown below

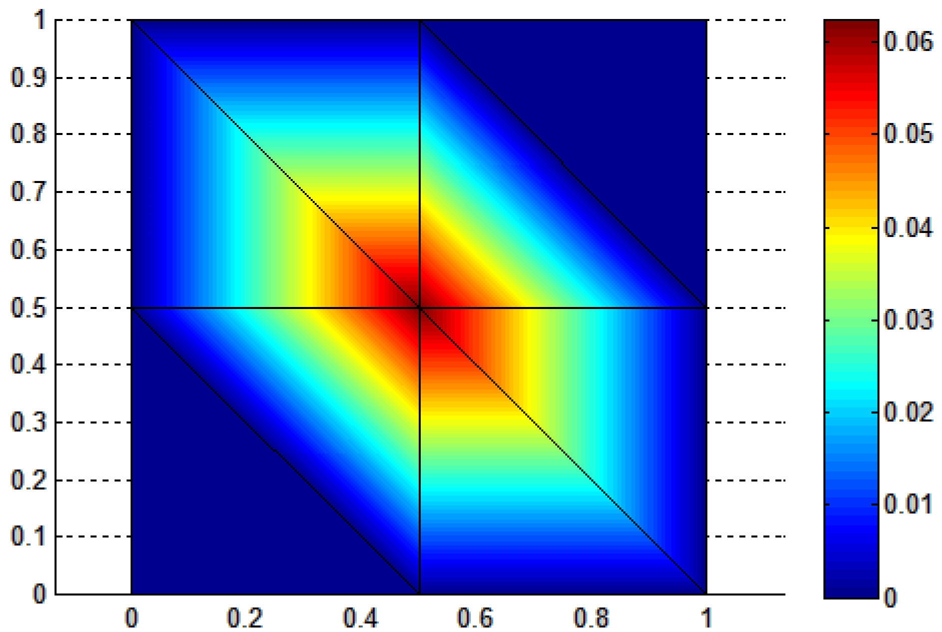
ans =

1.0000	-0.5000	0	-0.5000	0	0	0	0	0
-0.5000	2.0000	-0.5000	0	-1.0000	0	0	0	0
0	-0.5000	1.0000	0	0	-0.5000	0	0	0
-0.5000	0	0	2.0000	-1.0000	0	-0.5000	0	0
0	-1.0000	0	-1.0000	4.0000	-1.0000	0	-1.0000	0
0	0	-0.5000	0	-1.0000	2.0000	0	0	-0.5000
0	0	0	-0.5000	0	0	1.0000	-0.5000	0
0	0	0	0	-1.0000	0	-0.5000	2.0000	-0.5000
0	0	0	0	0	-0.5000	0	-0.5000	1.0000

F =

0.0417
0.1250
0.0833
0.1250
0.2500
0.1250
0.0833
0.1250
0.0417

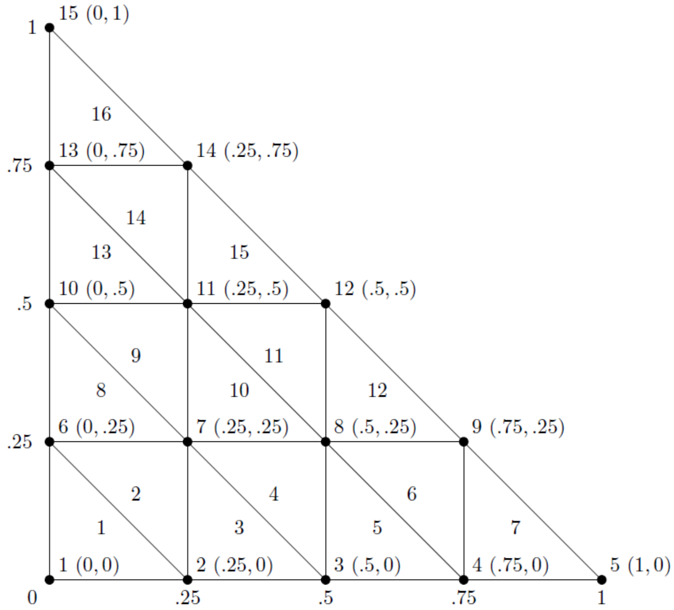
The 8 boundary nodes in b are shown above. When K is reduced to $Kb = 4$ and we get $4U = 0.25 \Rightarrow U = 0.0625$



Section 3.6 (Problem 6):

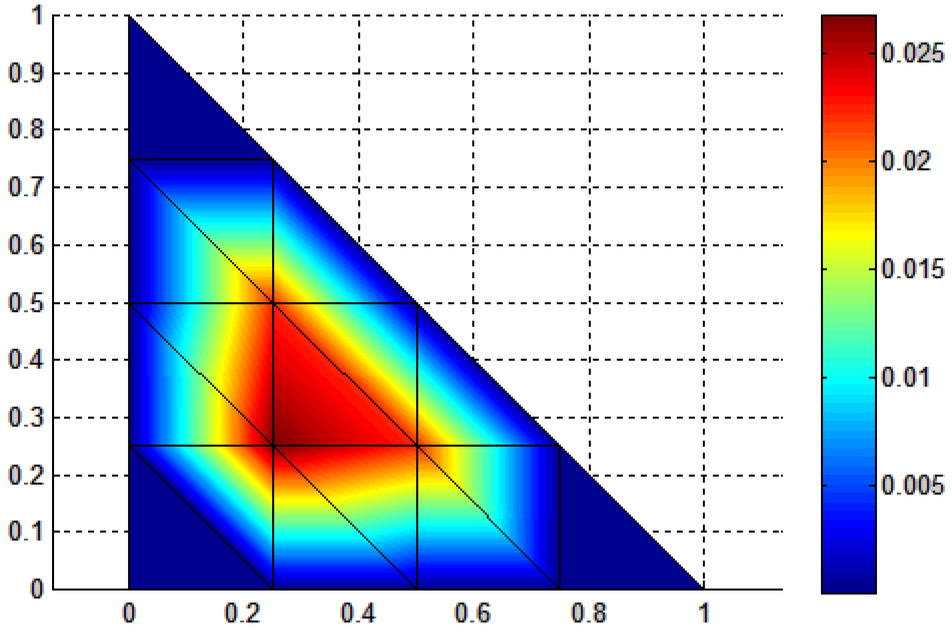
With $h = 1/4$, we have a mesh containing 16 triangles and 15 nodes (12 of them are boundary nodes) which are shown below along with p, t and b

$$p = \begin{bmatrix} 0 & 0 \\ .25 & 0 \\ .5 & 0 \\ .75 & 0 \\ 1 & 0 \\ 0 & .25 \\ .25 & .25 \\ .5 & .25 \\ .75 & .25 \\ 0 & .5 \\ .25 & .5 \\ .5 & .5 \\ 0 & .75 \\ .25 & .75 \\ 0 & 1 \end{bmatrix} \quad t = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 7 & 6 \\ 2 & 3 & 7 \\ 3 & 8 & 7 \\ 3 & 4 & 8 \\ 4 & 9 & 8 \\ 4 & 5 & 9 \\ 6 & 7 & 10 \\ 7 & 11 & 10 \\ 7 & 8 & 11 \\ 8 & 12 & 11 \\ 8 & 9 & 12 \\ 10 & 11 & 13 \\ 11 & 14 & 13 \\ 11 & 12 & 14 \\ 13 & 14 & 15 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 9 \\ 10 \\ 12 \\ 13 \\ 14 \\ 15 \end{bmatrix}$$



The output Kb, Fb and U is shown below

Kb =		Fb =	U =
(1,1)	1	0	0
(2,2)	1	0	0
(3,3)	1	0	0
(4,4)	1	0	0
(5,5)	1	0	0
(6,6)	1	0	0
(7,7)	4	0.0625	0.0268
(8,7)	-1	0.0625	0.0223
(11,7)	-1	0	0
(7,8)	-1	0	0
(8,8)	4	0.0625	0.0223
(9,9)	1	0	0
(10,10)	1	0	0
(7,11)	-1	0	0
(11,11)	4	0	0
(12,12)	1		
(13,13)	1		
(14,14)	1		
(15,15)	1		



<u>Section 3.6 (Problem 9):</u>

For $U = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2$, $\phi(x,y)$ which equals 1 at $x = y = 0$ and zero at all other nodes can be found by solving the following system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0.5 & 0 & 0.25 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 1 & 0.5 & 0.25 & 0.5 & 0.25 \\ 1 & 0.5 & 1 & 0.25 & 0.5 & 1 & 0.25 & 0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Using MATLAB's code shown below, the output of the code is shown below

```
F=[1;0;0;0;0;0;0;0];
Ph=[1 0 0  0   0   0   0   0;
    1 0 .5  0   0  .25  0   0;
    1 0  1  0   0   1   0   0;
    1 .5 0 .25  0   0   0   0;
    1 1  0  1   0   0   0   0;
    1 1 .5  1   .5 .25  .5 .25;
    1 .5 1 .25  .5  1  .25 .5;
    1 1  1  1   1   1   1   1];
a=inv(Ph)*F
```

```
a =

     1
    -3
    -3
     2
     5
     2
    -2
    -2
```

Section 3.6 (Problem 11):

(a) If $v_1 = (1,1), v_2 = (-1,1), v_3 = (-1,-1), v_4 = (1,-1)$ and $U(v_i) = U_i$ where $U = a + bx + cy + dxy$, then

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} U_1 + U_2 + U_3 + U_4 \\ U_1 - U_2 - U_3 + U_4 \\ U_1 + U_2 - U_3 - U_4 \\ U_1 - U_2 + U_3 - U_4 \end{bmatrix}$$

The inverse of the 4 by 4 matrix is obtained from MATLAB which is shown blow

```
EDU>> A=[1 1 1 1; 1 -1 1 -1; 1 -1 -1 1; 1 1 -1 -1]

A =

     1     1     1     1
     1    -1     1    -1
     1    -1    -1     1
     1     1    -1    -1

EDU>> inv(A)

ans =

    0.2500    0.2500    0.2500    0.2500
    0.2500   -0.2500   -0.2500    0.2500
    0.2500    0.2500   -0.2500   -0.2500
    0.2500   -0.2500    0.2500   -0.2500

EDU>> 4*inv(A)

ans =

     1     1     1     1
     1    -1    -1     1
     1     1    -1    -1
     1    -1     1    -1
```

(b+c) From part (a) we can deduce that

$$\begin{bmatrix} b \\ c \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = G\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad \Rightarrow \quad G\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } G\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, $[1; 1; 1; 1]$ and $[1; -1; 1; -1]$ are in the null space of G

Section 3.6 (Problem 18):

For two hat functions on the unit interval, with $U(0) = U(1) = 0$ and step size $h = 1/3$, the matrices K and M are

$$K = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \quad \text{and} \quad M = \frac{1}{18}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

$KU = \lambda MU \Rightarrow M^{-1}KU = \lambda U$, where $M^{-1}K$ is shown below

$$M^{-1}K = \frac{18}{15}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} = \frac{6}{5}\begin{bmatrix} 27 & -18 \\ -18 & 27 \end{bmatrix} = \frac{54}{5}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} = \frac{54}{5}N$$

The characteristic polynomial of the matrix N is $t^2 - 6t + 5$ whose roots are at $t = 1$ and $t = 5$. Therefore, lowest $\lambda = 54/5 = 10.8$ which is 'fairly' close to $\pi^2 \approx 9.87$.