18.085 Quiz 3 Professor Strang December 9, 2010

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2. 25
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100. ✓

Thank you for taking 18.085!

1. $f(x)$ is the 2π–periodic function that equals $e^{ix/2}$ from $x = 0$ to $x = 2\pi$.

   (a) What is the value of $f(x)$ at $x = 3\pi$? Is that the same as $e^{3\pi i/2}$?

   (b) Find the Fourier coefficients c_k in $f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$.

   (c) What is the decay rate of those coefficients c_k ? How is that decay rate explained using the smoothness properties of $f(x)$?

1) $f(x) = f(x+2\pi)$, $f(x) = e^{ix/2}$ for $0 \leq x < 2\pi$

ⓐ $f(3\pi) = f(\pi) = e^{i\frac{\pi}{2}}$

This is not the same as $e^{i\frac{3\pi}{2}}$, because $e^{i\frac{3\pi}{2}} = -i \neq e^{i\frac{\pi}{2}} = i$

ⓑ $f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx} \longrightarrow c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx$

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k-\frac{1}{2})x}\,dx = \frac{1}{2\pi}\left(\frac{-1}{i(k-\frac{1}{2})}\right)\left[e^{-i(k-\frac{1}{2})x}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{1}{i(k-\frac{1}{2})}\right)\left[1 - e^{-i(k-\frac{1}{2})2\pi}\right]$$

$$e^{-i(k-\frac{1}{2})2\pi} = e^{-i\overbrace{(2k-1)}^{odd}\pi} = -1 \quad \text{for } k \in \mathbb{Z}$$

$$\boxed{c_k = \frac{1}{i\pi(k-\frac{1}{2})}}$$

ⓒ The decay rate of c_k is $\frac{1}{k}$.

This is to be expected since there is a jump in $f(x)$ at $x = \dots, 0, 2\pi, 4\pi, \dots$ and a jump in $f(x)$ corresponds to a decay rate of $\frac{1}{k}$ in c_k.

2. (a) Set up 4 equations for the numbers a_0, a_1, a_2, a_3 so that the cubic $p(x)$ has the values b_0, b_1, b_2, b_3 at the 4 points $x = 1, i, -1, -i$:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \begin{matrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{matrix} \quad \text{at} \quad x = \begin{matrix} 1 \\ i \\ -1 \\ -i \end{matrix} .$$

(b) Those 4 equations could be written in matrix-vector form as $\mathrm{M}a = b$.
What is the inverse of that matrix M?

(c) Conclusion: Matching these values b_0, b_1, b_2, b_3 at these points is the same as

________________________ .

2) (a) i) $a_0 + a_1 + a_2 + a_3 = b_0$

ii) $a_0 + a_1 i - a_2 - a_3 i = b_1$

iii) $a_0 - a_1 + a_2 - a_3 = b_2$

iv) $a_0 - a_1 i + a_2 + a_3 i = b_3$

$$\Downarrow$$

(b)

$$Ma = b \quad \text{where} \quad M = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}}_{\text{" } F_4}, \quad a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Fourier Matrix

since $M = F_4$, $M^{-1} = F_4^{-1} = \frac{1}{4}\overline{F_4}^T = \boxed{\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} = M^{-1}}$

(c) Matching these b_0, b_1, b_2, b_3 at these points is the same as doing a discrete Fourier transform from $a \leftrightarrow$ to b, where a is in k-space and b is in x-space

3. (a) Find the discrete Fourier coefficients c_0, c_1, c_2 of the discrete values $f_0 = 2,\ f_1 = -1,\ f_2 = -1$.

(b) What is the cyclic convolution $c \circledast c$ of $c = (c_0, c_1, c_2)$ with itself ?

(c) Check the convolution rule by multiplying component by component to get $f .* f = (f_0^2, f_1^2, f_2^2)$ and finding the discrete coefficients of this vector. The result should be (and is) the same as ____________________ .

3) (a) $f_0 = 2,\ f_1 = -1,\ f_2 = -1$. Find $c_0; c_1, c_2$

$$f = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \qquad F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix} \quad \text{where } w = e^{i\frac{2\pi}{3}}$$

$$F_3^{-1} = \frac{1}{3}\overline{F_3}^T = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \overline{w} & \overline{w^2} \\ 1 & \overline{w^2} & \overline{w} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \quad \text{where } \omega = e^{-i\frac{2\pi}{3}}$$

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = F_3^{-1} f = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ (2-\omega-\omega^2) \\ (2-\omega^2-\omega) \end{bmatrix}$$

so $c_0 = 0$, $c_1 = c_2 = \frac{1}{3}(2 - w - w^2) = \frac{1}{3}(2 - e^{-i\frac{2\pi}{3}} - e^{-i\frac{4\pi}{3}}) = \frac{1}{3}(2 - e^{-i\frac{2\pi}{3}} - e^{i\frac{2\pi}{3}})$

$$= \frac{1}{3}\left(2 - 2\left[\frac{e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}}{2}\right]\right) = \frac{1}{3}\left(2 - 2\cos\left(\frac{2\pi}{3}\right)\right)$$

$$= \frac{1}{3}\left(2 - 2\left(-\frac{1}{2}\right)\right) = \frac{1}{3}(3) = 1$$

So, $\boxed{c_0 = 0, \quad c_1 = 1, \quad c_2 = 1}$ ✓

(b) $c \circledast c = (0, 1, 1) \circledast (0, 1, 1) = \boxed{(2, 1, 1)}$ ✓

↳ $C(z) = 0 + z + z^2 \qquad (C(z))^2 = (z + z^2)^2 = z^2 + 2z^3 + z^4 = z^2 + 2z^0 + z^1 \qquad$ (cyclic: $z^{3n} = z^n$)

(c) $F_3(c \circledast c) = (F_3 c) .* (F_3 c) = f .* f = (f_0^2, f_1^2, f_2^2) = (4, 1, 1)$ ✓

$$(c \circledast c) = F_3^{-1}(F_3(c \circledast c)) = F_3^{-1}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 4+\omega+\omega^2 \\ 4+\omega+\omega^2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$ ✓

$\left(4 + \omega + \omega^2 = 4 + 2\cos\left(\frac{2\pi}{3}\right) = 4 - 1 = 3\right)$ ✓

So the discrete coefficients of the vector $(f .* f)$ are identical to $(c \circledast c)$ as was expected.

4. A Helmholtz equation (with $c =$ real constant) looks like

$$-\frac{d^2u}{dx^2} + u(x) - c^2u(x) = f(x) \quad \text{for} \quad -\infty < x < \infty.$$

(a) With Fourier integrals, transform this into an equation for $\hat{u}(k)$.
Find $\widehat{u}(k)$ in terms of $\widehat{f}(k)$. Give a formula (an integral, not to compute it) for $u(x)$.

(b) Which values of c lead to difficulty in working with your formula for $\hat{u}(k)$?
What is the problem?

(c) With $c = 1$ and $f(x) = \delta(x)$, show directly that $-u''(x) = \delta(x)$ has no solution with $u(x) \to 0$ as $|x| \to \infty$.

4) $-\frac{d^2u}{dx^2} + u(x) - c^2 u(x) = f(x)$ for $-\infty < x < \infty$ with c = real constant

10/10 ⓐ $$-\int_{-\infty}^{\infty} \frac{d^2u}{dx^2} e^{-ikx} dx + (1-c^2)\int_{-\infty}^{\infty} u(x) e^{-ikx} dx = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$-(ik)^2 \hat{u}(k) + (1-c^2)\hat{u}(k) = \hat{f}(k)$$

$$(k^2+1-c^2)\hat{u}(k) = \hat{f}(k)$$

$$\boxed{\hat{u}(k) = \frac{\hat{f}(k)}{(1-c^2)+k^2}} = \hat{f}(k)\,\hat{g}(k) \quad \text{where } \hat{g}(k) \equiv \frac{1}{(1-c^2)+k^2}$$

$$u(x) = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt = \int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$

To find $g(x)$, we must find the inverse Fourier transform of $\hat{g}(k) = \frac{1}{(1-c^2)+k^2}$

$$\hat{g}(k) = \frac{1}{(1-c^2)+k^2} = \frac{1}{2\sqrt{1-c^2}}\,\frac{2\sqrt{1-c^2}}{(1-c^2)+k^2} = \frac{1}{2a}\,\frac{2a}{a^2+k^2} \quad \text{with } a = \sqrt{1-c^2}$$

This is the Fourier transform of a two-sided decaying pulse (assuming $a>0$).

$$g(x) = \frac{1}{2a}e^{-a|x|} = \frac{1}{2a}\begin{cases} e^{ax}, & x<0 \\ e^{-ax}, & x \geq 0 \end{cases}$$

so $$\boxed{u(x) = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt \quad \text{where } g(x) = \frac{1}{2a}e^{-a|x|} \text{ and } a = \sqrt{1-c^2}}$$

Excellent!

ⓑ 5/5 For $u(x)$ to be a solution, a must be a real, positive number.

This restricts c to $-1 < c < 1$.

If $|c| \geq 1$, there will be an issue with the solution $u(x)$, because then a will be 0 or imaginary, which will mean that $g(x)$ would not approach 0 as $|x| \to \infty$ (and thus $u(x)$ would also not approach 0 as $|x| \to \infty$)

10/10 ⓒ If $c = 1$, then from the criterion above there is no solution in which $u(x) \to 0$ as $|x| \to \infty$.

But, this can also be shown directly (with $f(x) = \delta(x)$):

$$-u''(x) + \cancel{u(x) - u(x)} = f(x) = \delta(x) \rightarrow u''(x) = -\delta(x) \rightarrow u'(x) = -S(x) + C_1$$

$$\rightarrow u(x) = -R(x) + C_1 x + C_2$$

$$u(x) = \begin{cases} (C_1 - 1)x + C_2, & x \geq 0 \\ C_1 x + C_2, & x \leq 0 \end{cases}$$

Clearly $u(x)$ cannot approach 0 for both $x \to \infty$ and $x \to -\infty$ no matter what values are chosen for C_1 and C_2.

Ex If $C_1 = 1, C_2 = 0$, $u(x) = 0$ for $x \geq 0$ but $u(x) \to -\infty$ for $x \to -\infty$

If $C_1 = 0, C_2 = 0$, $u(x) = 0$ for $x \leq 0$ but $u(x) \to -\infty$ for $x \to \infty$