4.1.2. The Fourier series of the square wave function is \( sw(x) = \frac{1}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \ldots \right) \), hence Parseval’s formula implies \( 2\pi = \int_{-\pi}^{\pi} |sw(x)|^2 dx = \pi \cdot \frac{1}{\pi^2} (1^2 + (\frac{1}{3})^2 + \ldots) \), hence \( 1 + \left( \frac{1}{3} \right)^2 + \left( \frac{1}{5} \right)^2 + \ldots = \frac{\pi^2}{8} \).

4.1.3. \( f \) is even, hence \( b_k = 0 \). We compute \( a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2} \), and for \( k > 0 \), \( a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)dx = \frac{2}{\pi} \int_{0}^{\pi/2} \cos(kx)dx = \frac{\pi}{\pi^2} \sin(k\pi) \).

4.1.4. By the orthogonality of sines and cosines we have \( a_0 = \frac{1}{\pi} \int_{-T/2}^{T/2} f(x)dx \), and for \( k > 0 \), \( a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos(\frac{2\pi kx}{T}) dx, b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin(\frac{2\pi kx}{T}) dx, c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(x)e^{-2\pi i kx/T} dx. \)

4.1.5. The decay rate is cubic, since the function is quadratic.

4.1.7. Since \( f \) is 0 on \([-\pi, 0]\), we just think of \( f \) as \( \sin x \) on \([0, \pi]\). Hence \( a_0 = \frac{1}{\pi} \int_{0}^{\pi} \sin x dx = \frac{1}{\pi} \), and for \( k > 0 \), \( a_k = \frac{1}{\pi} \int_{0}^{\pi} \sin kx dx = \frac{1}{\pi} \int_{0}^{\pi} \sin(k+1)x dx - \frac{1}{\pi} \int_{0}^{\pi} \sin(k-1)x dx \). Hence \( a_k = 0 \) for odd \( k \), and \( a_k = \frac{1}{\pi(k+1)} \cdot \frac{1}{\pi(k-1)} = \frac{1}{\pi(1-\pi^2)} \) for even \( k \).

Next, \( b_1 = \frac{1}{\pi} \int_{0}^{\pi} (\sin x)^2 dx = \frac{1}{2} \) and for \( k > 1 \), \( b_k = \frac{1}{\pi} \int_{0}^{\pi} \sin x \sin kx dx = 0 \). Therefore, the Fourier series expansion of \( f \) is

\[
 f(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \left( \frac{1}{1 \cdot 3} \cos 2x + \frac{1}{3 \cdot 5} \cos 4x + \frac{1}{5 \cdot 7} \cos 6x + \ldots \right).
\]

4.1.8. a) For \( u = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \) and \( v = (1, \frac{1}{3}, \frac{1}{5}, \ldots) \) we have \( u^T u = \sum_{k=0}^{\infty} (\frac{1}{k})^2 = \frac{\pi^2}{8} \) and \( u^T v = \sum_{k=0}^{\infty} \frac{1}{k^2} \cdot \frac{1}{k^2} = \frac{\pi^2}{8} \). Hence, the Cauchy-Schwarz inequality asserts that \((6/5)^2 \leq \frac{\pi^2}{8} \cdot \frac{\pi^2}{8} \).

b) This follows immediately from a) and Parseval’s identity.

4.1.9. The solution \( u(r, \theta) \) has the form \( u = a_0 + r a_1 \cos \theta + r b_1 \sin \theta + r^2 a_2 \cos 2\theta + r^2 b_2 \sin 2\theta + \ldots \). We compute \( a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta d\theta = 0, a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos k\theta d\theta = 0 \) and \( b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin k\theta d\theta = \frac{2 \cos k\pi}{k} \). Hence, \( u(r, \theta) = 2(r \sin \theta - r^2 \sin 2\theta + r^3 \sin 3\theta - \ldots) \). The Taylor series for \( 2 \log(1 + z) \) on the unit circle \( z = e^{i\theta} \) is \( 2 \log(1 + e^{i\theta}) = 2(e^{i\theta} - \frac{1}{2} e^{2i\theta} + \frac{1}{3} e^{3i\theta} - \ldots) \). Taking its imaginary part, we obtain exactly the above series of \( u(r, \theta) \) for \( r = 1 \).