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Answers to Problem Set 9	
TAs: Dorian Croitoru, Nan Li	

4.1.2. The Fourier series of the square wave function is $sw(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + ...\right)$, hence Parseval's formula implies $2\pi = \int_{-\pi}^{\pi} |sw(x)|^2 dx = \pi \cdot \frac{16}{\pi^2}(1^2 + (\frac{1}{3})^2 + ...)$, hence $1 + (\frac{1}{3})^2 + (\frac{1}{5})^2 + ... = \frac{\pi^2}{8}$.

4.1.3. f is even, hence $b_k = 0$. We compute $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2}$, and for $k > 0$, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{2}{\pi}\int_0^{\pi/2}\cos kx dx = \frac{2}{\pi k}\sin(\frac{\pi k}{2})$.

4.1.4. By the orthogonality of sines and cosines we have $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)dx$, and for $k > 0, a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\frac{2\pi kx}{T}dx, b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\frac{2\pi kx}{T}dx, c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-2\pi ikx/T}dx$.

4.1.5. The decay rate is cubic, since the function is quadratic.

4.1.7. Since f is 0 on $[-\pi, 0]$, we just think of f as $\sin x$ on $[0, \pi]$. Hence $a_0 = \frac{1}{2\pi}\int_0^{\pi}\sin x dx = \frac{1}{\pi}$, and for $k > 0, a_k = \frac{1}{\pi}\int_0^{\pi}\sin x\cos kx dx = \frac{1}{2\pi}\int_0^{\pi}\sin(k+1)x dx - \frac{1}{2\pi}\int_0^{\pi}\sin(k-1)x dx$. Hence $a_k = 0$ for odd k, and $a_k = \frac{1}{\pi(k+1)} - \frac{1}{\pi(k-1)} = \frac{2}{\pi(1-k^2)}$ for even k.

Next, $b_1 = \frac{1}{\pi}\int_0^{\pi}(\sin x)^2 dx = \frac{1}{2}$ and for $k > 1, b_k = \frac{1}{\pi}\int_0^{\pi}\sin x\sin kx dx = 0$. Therefore, the Fourier series expansion of f is

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{1}{1\cdot 3}\cos 2x + \frac{1}{3\cdot 5}\cos 4x + \frac{1}{5\cdot 7}\cos 6x + ...\right).$$

4.1.8. a) For $u = (1, \frac{1}{2}, \frac{1}{4}, ...)$ and $v = (1, \frac{1}{3}, \frac{1}{9}, ...)$ we have $u^T u = \sum_{k\geq 0}(\frac{1}{2^k})^2 = \frac{4}{3}, v^T v = \sum_{k\geq 0}(\frac{1}{3^k})^2 = \frac{9}{8}$ and $u^T v = \sum_{k\geq 0}\frac{1}{2^k}\cdot\frac{1}{3^k} = \frac{6}{5}$. Hence, the Cauchy-Schwartz inequality asserts that $(6/5)^2 \leq \frac{4}{3}\cdot\frac{9}{8}$.

b) This follows immediately from a) and Parseval's identity.

4.1.9. The solution $u(r, \theta)$ has the form $u = a_0 + ra_1\cos\theta + rb_1\sin\theta + r^2 a_2\cos 2\theta + r^2 b_2\sin 2\theta + ...$ We compute $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta d\theta = 0, a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\cos k\theta d\theta = 0$ and $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\sin k\theta d\theta = \frac{-2\cos k\pi}{k}$. Hence, $u(r, \theta) = 2(r\sin\theta - r^2\frac{\sin 2\theta}{2} + r^3\frac{\sin 3\theta}{3} - ...)$. The Taylor series for $2\log(1+z)$ on the unit circle $z = e^{i\theta}$ is $2\log(1 + e^{i\theta}) = 2(e^{i\theta} - \frac{1}{2}e^{2i\theta} + \frac{1}{3}e^{3i\theta} - ...)$. Taking its imaginary part, we obtain exactly the above series of $u(r, \theta)$ for $r = 1$.