18.085 Computational Science and Engineering	April 8, 2010
Answers to Problem Set 7	
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3.2.6. In general, the ϕ's that *don't satisfy* the boundary conditions should be dropped. (a) $\phi_0^d, \phi_0^s, \phi_3^d, \phi_3^s$ (b) ϕ_0^d, ϕ_3^d (c) ϕ_0^d, ϕ_0^s.

3.2.8. The displacement and slope functions centered at ih are (cf. page 245) $\phi_i^d = \left(\left|\frac{x-ih}{h}\right| - 1\right)^2 \left(2\left|\frac{x-ih}{h}\right|+1\right), \phi_i^s = \left(\left|\frac{x-ih}{h}\right| - 1\right)^2 \left|\frac{x-ih}{h}\right|$. Doing calculations, $(\phi_i^d)'' = \frac{12}{h^3}|x-ih| - \frac{6}{h^2}, (\phi_i^s)'' = \frac{6}{h^3}|x-ih| - \frac{4}{h^2}$. In our problem, $h = \frac{1}{3}, i = 0, 1$ so $(\phi_0^d)'' = 324|x| - 54, (\phi_1^d)'' = 324|x - \frac{1}{3}| - 54, (\phi_0^s)'' = 162|x| - 36, (\phi_1^s)'' = 162|x - \frac{1}{3}| - 36$.

3.2.9. We integrate the 16 pairwise products of $(\phi_0^d)'' = 324|x| - 54, (\phi_1^d)'' = 324|x - \frac{1}{3}| - 54, (\phi_0^s)'' = 162|x| - 36, (\phi_1^s)'' = 162|x - \frac{1}{3}| - 36$ from 0 to 1 to obtain

$$K_e = \begin{bmatrix} 324 & 162 & -324 & -162 \\ 162 & 108 & -162 & -54 \\ -324 & -162 & 648 & 162 \\ -162 & -54 & 162 & 216 \end{bmatrix}$$

Note that by symmetry reasons, K_e is symmetric, and its (3,3) entry is twice the (1,1) entry, the (4,4) entry is the same as (2,2), and (1,2) is same as (3,4) and opposite of (2,3).

3.2.10. By symmetry, we can easily assemble the symmetric 8x8 matrix K from K_e in problem 9:

$$K = \begin{bmatrix} 324 & 162 & -324 & -162 & 0 & 0 & 0 & 0 \\ 162 & 108 & -162 & -54 & 0 & 0 & 0 & 0 \\ -324 & -162 & 648 & 162 & -324 & -162 & 0 & 0 \\ -162 & -54 & 162 & 216 & -162 & -54 & 0 & 0 \\ 0 & 0 & -324 & -162 & 648 & 162 & -324 & -162 \\ 0 & 0 & -162 & -54 & 162 & 216 & -162 & -54 \\ 0 & 0 & 0 & 0 & -324 & -162 & 324 & 162 \\ 0 & 0 & 0 & 0 & -162 & -54 & 162 & 108 \end{bmatrix}$$

3.3.1. For $v = (1, 0) = w$, we solve $\frac{du}{dx} = 1, \frac{du}{dy} = 0$ to get $u(x, y) = x + C$ and $-\frac{ds}{dx} = 0, \frac{ds}{dy} = 1$ to get $s(x, y) = y + C$. Thus the equipotentials are vertical lines ($x = C$) and the streamlines are horizontal lines ($y = C$).

3.3.2. $(w_1, w_2) = (0, x)$ is not a gradient field, since $\frac{dw_1}{dy} = 0 \neq 1 = \frac{dw_2}{dx}$. The streamlines are vertical (since w always points in a vertical direction), the flow is downward on the left half of the plane, and upward on the right half (and 0 on the y axis), so there is a "rotation" taking place at infinity in some sense.

3.3.3. The flows out of nodes 1,2,4 are $w_1 + w_3, w_2 + w_4 - w_1, w_6 - w_3$. The flow across the dashed line is the (signed) sum of flows of the edges crossing the line $= w_2 + w_4 + w_6$.

3.3.4. The circulation around the left rectangle is $w_3 + w_6 - w_4 - w_1$. Around the right rectangle: $w_4 + w_7 - w_5 - w_2$. Around the large rectangle: $w_3 + w_6 + w_7 - w_1 - w_2 - w_5$.

3.3.5. For $v_1 = -y, v_2 = 0$ Stoke's law becomes $\int_C -y dx = \int_C v_1 dx + v_2 dy = \int\int_R \left(\frac{dv_2}{dx} - \frac{dv_1}{dy} \right) dx dy = \int\int_R 1 dx dy = Area(R)$. The ellipse is parametrized by $x = a\cos t, y = b\sin t, 0 \leq t \leq 2\pi$. Hence its area is $\int_C -y dx = \int_0^{2\pi} -(b\sin t)(-a\sin t) dt = \pi ab$.

3.3.6. For $v = (y^2, x^2)$, the curl is $2x - 2y \neq 0$ hence v is not a gradient field. The curl of $v = (y^2, 2xy)$ is 0, so we can solve $\frac{du}{dx} = y^2$, $\frac{du}{dy} = 2xy$ to get $u(x, y) = xy^2 + C$.

3.3.9. (a) $u(x, y) = y$ gives horizontal lines. So the streamlines are vertical lines, i.e. $s(x, y) = x$ works. (b) $u = x - y$ gives lines at an angle of $\pi/4$ with the x axis ($x - y = C$). So the streamlines are lines perpendicular to $x - y = 0$, i.e. $s(x, y) = x + y$. (c) $u(x, y) = \log(x^2 + y^2)^{1/2}$ gives concentric circles centered at the origin ($x^2 + y^2 = C$). So the streamlines are lines passing through the origin, i.e. $s(x, y) = y/x$ (or $\arctan \frac{y}{x}$) works.